

# Permutation Characters in GAP

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## Abstract

This is a loose collection of examples of computations with permutation characters and possible permutation characters in the GAP system [GAP04]. We mainly use the GAP implementation of the algorithms to compute possible permutation characters that are described in [BP98], and information from the Atlas of Finite Groups [CCN<sup>+</sup>85].

A **possible permutation character** of a finite group  $G$  is a character satisfying the conditions listed in Section “Possible Permutation Characters” of the GAP Reference Manual.

(Sections 14 and 15 were added in October 2001, Section 16.1 was added in June 2009, Section 16.2 was added in September 2009, Section 16.3 was added in October 2009, and Section 16.4 was added in November 2009.)

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In the following, the GAP Character Table Library [Bre12] will be used frequently.

```
gap> LoadPackage( "ctbllib", "1.2" );
true
```

## 1 Some Computations with $M_{24}$

We start with the sporadic simple Mathieu group  $G = M_{24}$  in its natural action on 24 points.

```
gap> g:= MathieuGroup( 24 );;
gap> SetName( g, "m24" );
gap> Size( g ); IsSimple( g ); NrMovedPoints( g );
244823040
true
24
```

The conjugacy classes that are computed for a group can be ordered differently in different GAP sessions. In order to make the output shown in the following examples stable, we first sort the conjugacy classes of  $G$  for our purposes.

```
gap> ccl:= AttributeValueNotSet( ConjugacyClasses, g );;
gap> HasConjugacyClasses( g );
false
gap> invariants:= List( ccl, c -> [ Order( Representative( c ) ),
> Size( c ), Size( ConjugacyClass( g, Representative( c )^2 ) ) ] );;
gap> SortParallel( invariants, ccl );
gap> SetConjugacyClasses( g, ccl );
```

The permutation character  $\pi$  of  $G$  corresponding to the action on the moved points is constructed. This action is 5-transitive.

```
gap> NrConjugacyClasses( g );
26
gap> pi:= NaturalCharacter( g );
Character( CharacterTable( m24 ), [ 24, 8, 0, 6, 0, 0, 4, 0, 4, 2, 0, 3, 3,
2, 0, 2, 0, 0, 1, 1, 1, 1, 0, 0, 1, 1 ] )
```

```

gap> IsTransitive( pi ); Transitivity( pi );
true
5
gap> Display( pi );
CT1

```

2	10	10	9	3	3	7	7	5	2	3	3	1	1	4	2	.	2	2	1	1	.
3	3	1	1	3	2	1	.	1	1	1	1	1	1	.	.	.	1	1	.	.	1
5	1	.	1	1	.	.	.	.	1	.	.	.	.	.	1	.	.	.	.	.	1
7	1	1	.	.	1	.	.	.	.	.	.	1	1	.	.	.	.	.	1	1	.
11	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.
23	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.

1a	2a	2b	3a	3b	4a	4b	4c	5a	6a	6b	7a	7b	8a	10a	11a	12a	12b	14a	14b	15a
----	----	----	----	----	----	----	----	----	----	----	----	----	----	-----	-----	-----	-----	-----	-----	-----

Y.1	24	8	.	6	.	.	4	.	4	2	.	3	3	2	.	2	.	.	1	1	1
-----	----	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---

2	.	.	.	.	.
3	1	1	1	.	.
5	1	.	.	.	.
7	.	1	1	.	.
11	.	.	.	.	.
23	.	.	.	1	1

15b	21a	21b	23a	23b
-----	-----	-----	-----	-----

Y.1	1	.	.	1	1
-----	---	---	---	---	---

pi determines the permutation characters of the  $G$ -actions on related sets, for example  $\text{piop}$  on the set of ordered and  $\text{piup}$  on the set of unordered pairs of points.

```

gap> piop:= pi * pi;
Character( CharacterTable( m24 ), [ 576, 64, 0, 36, 0, 0, 16, 0, 16, 4, 0, 9,
9, 4, 0, 4, 0, 0, 1, 1, 1, 1, 0, 0, 1, 1 ] )
gap> IsTransitive( piop );
false
gap> piup:= SymmetricParts( UnderlyingCharacterTable(pi), [ pi ], 2 )[1];
Character( CharacterTable( m24 ), [ 300, 44, 12, 21, 0, 4, 12, 0, 10, 5, 0,
6, 6, 4, 2, 3, 1, 0, 2, 2, 1, 1, 0, 0, 1, 1 ] )
gap> IsTransitive( piup );
false

```

Clearly the action on unordered pairs is not transitive, since the pairs  $[i, i]$  form an orbit of their own. There are exactly two  $G$ -orbits on the unordered pairs, hence the  $G$ -action on 2-sets of points is transitive.

```

gap> ScalarProduct( piup, TrivialCharacter( g ) );
2
gap> comb:= Combinations( [ 1 .. 24 ], 2 );
gap> hom:= ActionHomomorphism( g, comb, OnSets );
gap> pihom:= NaturalCharacter( hom );
Character( CharacterTable( m24 ), [ 276, 36, 12, 15, 0, 4, 8, 0, 6, 3, 0, 3,
3, 2, 2, 1, 1, 0, 1, 1, 0, 0, 0, 0, 0 ] )
gap> Transitivity( pihom );
1

```

In terms of characters, the permutation character `pihom` is the difference of `piup` and `pi`. Note that GAP does not know that this difference is in fact a character; in general this question is not easy to decide without knowing the irreducible characters of  $G$ , and up to now GAP has not computed the irreducibles.

```
gap> pi2s:= piup - pi;
VirtualCharacter( CharacterTable( m24 ), [ 276, 36, 12, 15, 0, 4, 8, 0, 6, 3,
  0, 3, 3, 2, 2, 1, 1, 0, 1, 1, 0, 0, 0, 0, 0, 0 ] )
gap> pi2s = pihom;
true
gap> HasIrr( g ); HasIrr( CharacterTable( g ) );
false
false
```

The point stabilizer in the action on 2-sets is in fact a maximal subgroup of  $G$ , which is isomorphic to the automorphism group  $M_{22} : 2$  of the Mathieu group  $M_{22}$ . Thus this permutation action is primitive. But we cannot apply `IsPrimitive` to the character `pihom` for getting this answer because primitivity of characters is defined in a different way, cf. `IsPrimitiveCharacter` in the GAP Reference Manual.

```
gap> IsPrimitive( g, comb, OnSets );
true
```

We could also have computed the transitive permutation character of degree 276 using the GAP Character Table Library instead of the group  $G$ , since the character tables of  $G$  and all its maximal subgroups are available, together with the class fusions of the maximal subgroups into  $G$ .

```
gap> tbl:= CharacterTable( "M24" );
CharacterTable( "M24" )
gap> maxes:= Maxes( tbl );
[ "M23", "M22.2", "2^4:a8", "M12.2", "2^6:3.s6", "L3(4).3.2_2",
  "2^6:(psl(3,2)xs3)", "L2(23)", "L3(2)" ]
gap> s:= CharacterTable( maxes[2] );
CharacterTable( "M22.2" )
gap> TrivialCharacter( s )^tbl;
Character( CharacterTable( "M24" ), [ 276, 36, 12, 15, 0, 4, 8, 0, 6, 3, 0,
  3, 3, 2, 2, 1, 1, 0, 1, 1, 0, 0, 0, 0, 0, 0 ] )
```

Note that the sequence of conjugacy classes in the library table of  $G$  does in general not agree with the succession computed for the group.

## 2 All Possible Permutation Characters of $M_{11}$

We compute all possible permutation characters of the Mathieu group  $M_{11}$ , using the three different strategies available in GAP.

First we try the algorithm that enumerates all candidates via solving a system of inequalities, which is described in [BP98, Section 3.2].

```
gap> m11:= CharacterTable( "M11" );;
gap> SetName( m11, "m11" );
gap> perms:= PermChars( m11 );
[ Character( m11, [ 1, 1, 1, 1, 1, 1, 1, 1, 1, 1 ] ),
  Character( m11, [ 11, 3, 2, 3, 1, 0, 1, 1, 0, 0 ] ),
  Character( m11, [ 12, 4, 3, 0, 2, 1, 0, 0, 1, 1 ] ),
```

```

Character( m11, [ 22, 6, 4, 2, 2, 0, 0, 0, 0, 0 ] ),
Character( m11, [ 55, 7, 1, 3, 0, 1, 1, 1, 0, 0 ] ),
Character( m11, [ 66, 10, 3, 2, 1, 1, 0, 0, 0, 0 ] ),
Character( m11, [ 110, 6, 2, 2, 0, 0, 2, 2, 0, 0 ] ),
Character( m11, [ 110, 6, 2, 6, 0, 0, 0, 0, 0, 0 ] ),
Character( m11, [ 110, 14, 2, 2, 0, 2, 0, 0, 0, 0 ] ),
Character( m11, [ 132, 12, 6, 0, 2, 0, 0, 0, 0, 0 ] ),
Character( m11, [ 144, 0, 0, 0, 4, 0, 0, 0, 1, 1 ] ),
Character( m11, [ 165, 13, 3, 1, 0, 1, 1, 1, 0, 0 ] ),
Character( m11, [ 220, 4, 4, 0, 0, 4, 0, 0, 0, 0 ] ),
Character( m11, [ 220, 12, 4, 4, 0, 0, 0, 0, 0, 0 ] ),
Character( m11, [ 220, 20, 4, 0, 0, 2, 0, 0, 0, 0 ] ),
Character( m11, [ 330, 2, 6, 2, 0, 2, 0, 0, 0, 0 ] ),
Character( m11, [ 330, 18, 6, 2, 0, 0, 0, 0, 0, 0 ] ),
Character( m11, [ 396, 12, 0, 4, 1, 0, 0, 0, 0, 0 ] ),
Character( m11, [ 440, 8, 8, 0, 0, 2, 0, 0, 0, 0 ] ),
Character( m11, [ 440, 24, 8, 0, 0, 0, 0, 0, 0, 0 ] ),
Character( m11, [ 495, 15, 0, 3, 0, 0, 1, 1, 0, 0 ] ),
Character( m11, [ 660, 4, 3, 4, 0, 1, 0, 0, 0, 0 ] ),
Character( m11, [ 660, 12, 3, 0, 0, 3, 0, 0, 0, 0 ] ),
Character( m11, [ 660, 12, 12, 0, 0, 0, 0, 0, 0, 0 ] ),
Character( m11, [ 660, 28, 3, 0, 0, 1, 0, 0, 0, 0 ] ),
Character( m11, [ 720, 0, 0, 0, 0, 0, 0, 0, 5, 5 ] ),
Character( m11, [ 792, 24, 0, 0, 2, 0, 0, 0, 0, 0 ] ),
Character( m11, [ 880, 0, 16, 0, 0, 0, 0, 0, 0, 0 ] ),
Character( m11, [ 990, 6, 0, 2, 0, 0, 2, 2, 0, 0 ] ),
Character( m11, [ 990, 6, 0, 6, 0, 0, 0, 0, 0, 0 ] ),
Character( m11, [ 990, 30, 0, 2, 0, 0, 0, 0, 0, 0 ] ),
Character( m11, [ 1320, 8, 6, 0, 0, 2, 0, 0, 0, 0 ] ),
Character( m11, [ 1320, 24, 6, 0, 0, 0, 0, 0, 0, 0 ] ),
Character( m11, [ 1584, 0, 0, 0, 4, 0, 0, 0, 0, 0 ] ),
Character( m11, [ 1980, 12, 0, 4, 0, 0, 0, 0, 0, 0 ] ),
Character( m11, [ 1980, 36, 0, 0, 0, 0, 0, 0, 0, 0 ] ),
Character( m11, [ 2640, 0, 12, 0, 0, 0, 0, 0, 0, 0 ] ),
Character( m11, [ 3960, 24, 0, 0, 0, 0, 0, 0, 0, 0 ] ),
Character( m11, [ 7920, 0, 0, 0, 0, 0, 0, 0, 0, 0 ] ) ]
gap> Length( perms );
39

```

Next we try the improved combinatorial approach that is sketched at the end of Section 3.2 in [BP98]. We get the same characters, except that they may be ordered in a different way; thus we compare the ordered lists.

```

gap> degrees:= DivisorsInt( Size( m11 ) );
gap> perms2:= [];
gap> for d in degrees do
>   Append( perms2, PermChars( m11, d ) );
>   od;
gap> Set( perms ) = Set( perms2 );
true

```

Finally, we try the algorithm that is based on Gaussian elimination and that is described in [BP98, Section 3.3].

```

gap> perms3:= [];

```

```

gap> for d in degrees do
>   Append( perms3, PermChars( m11, rec( torso:= [ d ] ) ) );
>   od;
gap> Set( perms ) = Set( perms3 );
true

```

GAP provides two more functions to test properties of permutation characters. The first one yields no new information in our case, but the second excludes one possible permutation character; note that `TestPerm5` needs a  $p$ -modular Brauer table, and the GAP character table library contains all Brauer tables of  $M_{11}$ .

```

gap> newperms:= TestPerm4( m11, perms );;
gap> newperms = perms;
true
gap> newperms:= TestPerm5( m11, perms, m11 mod 11 );;
gap> newperms = perms;
false
gap> Difference( perms, newperms );
[ Character( m11, [ 220, 4, 4, 0, 0, 4, 0, 0, 0 ] ) ]

```

GAP knows the table of marks of  $M_{11}$ , from which the permutation characters can be extracted. It turns out that  $M_{11}$  has 39 conjugacy classes of subgroups but only 36 different permutation characters, so three candidates computed above are in fact not permutation characters.

```

gap> tom:= TableOfMarks( "M11" );
TableOfMarks( "M11" )
gap> trueperms:= PermCharsTom( m11, tom );;
gap> Length( trueperms ); Length( Set( trueperms ) );
39
36
gap> Difference( perms, trueperms );
[ Character( m11, [ 220, 4, 4, 0, 0, 4, 0, 0, 0 ] ),
  Character( m11, [ 660, 4, 3, 4, 0, 1, 0, 0, 0 ] ),
  Character( m11, [ 660, 12, 3, 0, 0, 3, 0, 0, 0 ] ) ]

```

### 3 The Action of $U_6(2)$ on the Cosets of $M_{22}$

We are interested in the permutation character of  $U_6(2)$  (see [CCN<sup>+</sup>85, p. 115]) that corresponds to the action on the cosets of a  $M_{22}$  subgroup (see [CCN<sup>+</sup>85, p. 39]). The character tables of both the group and the point stabilizer are available in the GAP character table library, so we can compute class fusion and permutation character directly; note that if the class fusion is not stored on the table of the subgroup, in general one will not get a unique fusion but only a list of candidates for the fusion.

```

gap> u62:= CharacterTable( "U6(2)" );;
gap> m22:= CharacterTable( "M22" );;
gap> fus:= PossibleClassFusions( m22, u62 );
[ [ 1, 3, 7, 10, 14, 15, 22, 24, 24, 26, 33, 34 ],
  [ 1, 3, 7, 10, 14, 15, 22, 24, 24, 26, 34, 33 ],
  [ 1, 3, 7, 11, 14, 15, 22, 24, 24, 27, 33, 34 ],
  [ 1, 3, 7, 11, 14, 15, 22, 24, 24, 27, 34, 33 ],
  [ 1, 3, 7, 12, 14, 15, 22, 24, 24, 28, 33, 34 ],
  [ 1, 3, 7, 12, 14, 15, 22, 24, 24, 28, 34, 33 ] ]
gap> RepresentativesFusions( m22, fus, u62 );
[ [ 1, 3, 7, 10, 14, 15, 22, 24, 24, 26, 33, 34 ] ]

```

We see that there are six possible class fusions that are equivalent under table automorphisms of  $U_6(2)$  and  $M_{22}$ .

```

gap> cand:= Set( List( fus,
> x -> Induced( m22, u62, [ TrivialCharacter( m22 ) ], x )[1] ) );
[ Character( CharacterTable( "U6(2)" ), [ 20736, 0, 384, 0, 0, 0, 54, 0, 0,
0, 0, 48, 0, 16, 6, 0, 0, 0, 0, 0, 0, 0, 6, 0, 2, 0, 0, 0, 4, 0, 0, 0, 0,
1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ] ),
Character( CharacterTable( "U6(2)" ), [ 20736, 0, 384, 0, 0, 0, 54, 0, 0,
0, 48, 0, 0, 16, 6, 0, 0, 0, 0, 0, 0, 0, 6, 0, 2, 0, 0, 4, 0, 0, 0, 0, 0,
1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ] ),
Character( CharacterTable( "U6(2)" ), [ 20736, 0, 384, 0, 0, 0, 54, 0, 0,
48, 0, 0, 0, 16, 6, 0, 0, 0, 0, 0, 0, 0, 6, 0, 2, 0, 0, 4, 0, 0, 0, 0, 0, 0,
1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ] ) ]
gap> PermCharInfo( u62, cand ).ATLAS;
[ "1a+22a+252a+616a+1155c+1386a+8064a+9240c",
"1a+22a+252a+616a+1155b+1386a+8064a+9240b",
"1a+22a+252a+616a+1155a+1386a+8064a+9240a" ]
gap> aut:= AutomorphismsOfTable( u62 );; Size( aut );
24
gap> elms:= Filtered( Elements( aut ), x -> Order( x ) = 3 );
[ (10,11,12)(26,27,28)(40,41,42), (10,12,11)(26,28,27)(40,42,41) ]
gap> Position( cand, Permuted( cand[1], elms[1] ) );
3
gap> Position( cand, Permuted( cand[3], elms[1] ) );
2

```

The six fusions induce three different characters, they are conjugate under the action of the unique subgroup of order 3 in the group of table automorphisms of  $U_6(2)$ . The table automorphisms of order 3 are induced by group automorphisms of  $U_6(2)$  (see [CCN<sup>+</sup>85, p. 120]). As can be seen from the list of maximal subgroups of  $U_6(2)$  in [CCN<sup>+</sup>85, p. 115], the three induced characters are in fact permutation characters which belong to the three classes of maximal subgroups of type  $M_{22}$  in  $U_6(2)$ , which are permuted by an outer automorphism of order 3.

Now we want to compute the extension of the above permutation character to the group  $U_6(2).2$ , which corresponds to the action of this group on the cosets of a  $M_{22}.2$  subgroup.

```

gap> u622:= CharacterTable( "U6(2).2" );;
gap> m222:= CharacterTable( "M22.2" );;
gap> fus:= PossibleClassFusions( m222, u622 );
[ [ 1, 3, 7, 10, 13, 14, 20, 22, 22, 24, 29, 38, 39, 42, 41, 46, 50, 53, 58,
59, 59 ] ]
gap> cand:= Induced( m222, u622, [ TrivialCharacter( m222 ) ], fus[1] );
[ Character( CharacterTable( "U6(2).2" ), [ 20736, 0, 384, 0, 0, 0, 54, 0, 0,
48, 0, 0, 16, 6, 0, 0, 0, 0, 0, 6, 0, 2, 0, 4, 0, 0, 0, 0, 1, 0, 0, 0,
0, 0, 0, 0, 0, 1080, 72, 0, 48, 8, 0, 0, 0, 18, 0, 0, 0, 8, 0, 0, 2, 0,
0, 0, 0, 2, 2, 0, 0, 0, 0, 0, 0 ] ) ]
gap> PermCharInfo( u622, cand ).ATLAS;
[ "1a+22a+252a+616a+1155a+1386a+8064a+9240a" ]

```

We see that for the embedding of  $M_{22}.2$  into  $U_6(2).2$ , the class fusion is unique, so we get a unique extension of one of the above permutation characters. This implies that exactly one class of maximal subgroups of type  $M_{22}$  extends to  $M_{22}.2$  in a given group  $U_6(2).2$ .

## 4 Degree 20736 Permutation Characters of $U_6(2)$

Now we show an alternative way to compute the characters dealt with in the previous example. This works also if the character table of the point stabilizer is not available. In this situation we can compute all those characters that have certain properties of permutation characters.

Of course this may take much longer than the above computations, which needed only a few seconds. (The following calculations may need several hours, depending on the computer used.)

```
gap> cand:= PermChars( u62, rec( torso := [ 20736 ] ) );
[ Character( CharacterTable( "U6(2)" ), [ 20736, 0, 384, 0, 0, 0, 54, 0, 0,
  0, 0, 48, 0, 16, 6, 0, 0, 0, 0, 0, 0, 0, 6, 0, 2, 0, 0, 0, 0, 4, 0, 0, 0, 0,
  1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ] ),
  Character( CharacterTable( "U6(2)" ), [ 20736, 0, 384, 0, 0, 0, 54, 0, 0,
  0, 48, 0, 0, 16, 6, 0, 0, 0, 0, 0, 0, 0, 6, 0, 2, 0, 0, 0, 4, 0, 0, 0, 0, 0,
  1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ] ),
  Character( CharacterTable( "U6(2)" ), [ 20736, 0, 384, 0, 0, 0, 54, 0, 0,
  48, 0, 0, 0, 16, 6, 0, 0, 0, 0, 0, 0, 0, 6, 0, 2, 0, 0, 4, 0, 0, 0, 0, 0, 0,
  1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ] ) ]
```

For the next step, that is, the computation of the extension of the permutation character to  $U_6(2).2$ , we may use the above information, since the values on the inner classes are prescribed.

The question which of the three candidates for  $U_6(2)$  extends to  $U_6(2).2$  depends on the choice of the class fusion of  $U_6(2)$  into  $U_6(2).2$ . With respect to the class fusion that is stored on the GAP library table, the third candidate extends, as can be seen from the fact that this one is invariant under the permutation of conjugacy classes of  $U_6(2)$  that is induced by the action of the chosen supergroup  $U_6(2).2$ .

```
gap> u622:= CharacterTable( "U6(2).2" );;
gap> inv:= InverseMap( GetFusionMap( u62, u622 ) );
[ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, [ 11, 12 ], 13, 14, 15, [ 16, 17 ], 18, 19,
  20, 21, 22, 23, 24, 25, 26, [ 27, 28 ], [ 29, 30 ], 31, 32, [ 33, 34 ],
  [ 35, 36 ], 37, [ 38, 39 ], 40, [ 41, 42 ], 43, 44, [ 45, 46 ] ]
gap> ext:= List( cand, x -> CompositionMaps( x, inv ) );
[ [ 20736, 0, 384, 0, 0, 0, 54, 0, 0, 0, [ 0, 48 ], 0, 16, 6, 0, 0, 0, 0, 0,
  6, 0, 2, 0, 0, [ 0, 4 ], 0, 0, 0, 1, 0, 0, 0, 0, 0, 0 ],
  [ 20736, 0, 384, 0, 0, 0, 54, 0, 0, 0, [ 0, 48 ], 0, 16, 6, 0, 0, 0, 0, 0,
  6, 0, 2, 0, 0, [ 0, 4 ], 0, 0, 0, 1, 0, 0, 0, 0, 0, 0 ],
  [ 20736, 0, 384, 0, 0, 0, 54, 0, 0, 48, 0, 0, 16, 6, 0, 0, 0, 0, 6, 0,
  2, 0, 4, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0 ] ]
gap> cand:= PermChars( u622, rec( torso:= ext[3] ) );
[ Character( CharacterTable( "U6(2).2" ), [ 20736, 0, 384, 0, 0, 0, 54, 0, 0,
  48, 0, 0, 16, 6, 0, 0, 0, 0, 0, 6, 0, 2, 0, 4, 0, 0, 0, 0, 1, 0, 0, 0,
  0, 0, 0, 0, 0, 1080, 72, 0, 48, 8, 0, 0, 0, 18, 0, 0, 0, 8, 0, 0, 2, 0,
  0, 0, 0, 2, 2, 0, 0, 0, 0, 0, 0 ] ) ]
```

## 5 Degree 57572775 Permutation Characters of $O_8^+(3)$

The group  $O_8^+(3)$  (see [CCN<sup>+</sup>85, p. 140]) contains a subgroup of type  $2^{3+6}.L_3(2)$ , which extends to a maximal subgroup  $U$  in  $O_8^+(3).3$ . For the computation of the permutation character, we cannot use explicit induction since the table of  $U$  is not available in the GAP table library.

Since  $U \cap O_8^+(3)$  is contained in a  $O_8^+(2)$  subgroup of  $O_8^+(3)$ , we can try to find the permutation character of  $O_8^+(2)$  corresponding to the action on the cosets of  $U \cap O_8^+(3)$ , and then induce this character to  $O_8^+(3)$ .

This kind of computations becomes more difficult with increasing degree, so we try to reduce the problem further. In fact, the  $2^{3+6}.L_3(2)$  group is contained in a  $2^6 : A_8$  subgroup of  $O_8^+(2)$ , in which the index is only 15; the unique possible permutation character of this degree can be read off immediately.

Induction to  $O_8^+(3)$  through the chain of subgroups is possible provided the class fusions are available. There are 24 possible fusions from  $O_8^+(2)$  into  $O_8^+(3)$ , which are all equivalent w.r.t. table automorphisms of  $O_8^+(3)$ . If we later want to consider the extension of the permutation character in question to  $O_8^+(3).3$  then we have to choose a fusion of an  $O_8^+(2)$  subgroup that does **not** extend to  $O_8^+(2).3$ . But if for example our question is just whether the resulting permutation character is multiplicity-free then this can be decided already from the permutation character of  $O_8^+(3)$ .

```
gap> o8p3:= CharacterTable("O8+(3)");;
gap> Size( o8p3 ) / (2^9*168);
57572775
gap> o8p2:= CharacterTable( "O8+(2)" );;
gap> fus:= PossibleClassFusions( o8p2, o8p3 );;
gap> Length( fus );
24
gap> rep:= RepresentativesFusions( o8p2, fus, o8p3 );
[ [ 1, 5, 2, 3, 4, 5, 7, 8, 12, 16, 17, 19, 23, 20, 21, 22, 23, 24, 25, 26,
    37, 38, 42, 31, 32, 36, 49, 52, 51, 50, 43, 44, 45, 53, 55, 56, 57, 71,
    71, 71, 72, 73, 74, 78, 79, 83, 88, 89, 90, 94, 100, 101, 105 ] ]
gap> fus:= rep[1];;
gap> Size( o8p2 ) / (2^9*168);
2025
gap> sub:= CharacterTable( "2^6:A8" );;
gap> subfus:= GetFusionMap( sub, o8p2 );
[ 1, 3, 2, 2, 4, 5, 6, 13, 3, 6, 12, 13, 14, 7, 21, 24, 11, 30, 29, 31, 13,
  17, 15, 16, 14, 17, 36, 37, 18, 41, 24, 44, 48, 28, 33, 32, 34, 35, 35, 51,
  51 ]
gap> fus:= CompositionMaps( fus, subfus );
[ 1, 2, 5, 5, 3, 4, 5, 23, 2, 5, 19, 23, 20, 7, 37, 31, 17, 50, 51, 43, 23,
  23, 21, 22, 20, 23, 56, 57, 24, 72, 31, 78, 89, 52, 45, 44, 53, 55, 55,
  100, 100 ]
gap> Size( sub ) / (2^9*168);
15
gap> List( Irr( sub ), Degree );
[ 1, 7, 14, 20, 21, 21, 21, 28, 35, 45, 45, 56, 64, 70, 28, 28, 35, 35, 35,
  35, 70, 70, 70, 70, 140, 140, 140, 140, 140, 210, 210, 252, 252, 280, 280,
  315, 315, 315, 315, 420, 448 ]
gap> cand:= PermChars( sub, 15 );
[ Character( CharacterTable( "2^6:A8" ), [ 15, 15, 15, 7, 7, 7, 7, 7, 3, 3,
  3, 3, 3, 0, 0, 0, 3, 3, 3, 3, 3, 3, 3, 3, 1, 1, 1, 1, 0, 0, 0, 0, 0, 1,
  1, 1, 1, 1, 1, 0, 0 ] ) ]
gap> ind:= Induced( sub, o8p3, cand, fus );
[ Character( CharacterTable( "O8+(3)" ), [ 57572775, 59535, 59535, 59535,
  3591, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2187, 0, 27, 135, 135, 135, 243,
  0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 27, 27, 27, 0,
  0, 0, 0, 27, 27, 27, 27, 0, 8, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
  0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
  0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ] ) ]
gap> o8p33:= CharacterTable( "O8+(3).3" );;
gap> inv:= InverseMap( GetFusionMap( o8p3, o8p33 ) );
[ 1, [ 2, 3, 4 ], 5, 6, [ 7, 8, 9 ], [ 10, 11, 12 ], 13, [ 14, 15, 16 ], 17,
```

```

18, 19, [ 20, 21, 22 ], 23, [ 24, 25, 26 ], [ 27, 28, 29 ], 30,
[ 31, 32, 33 ], [ 34, 35, 36 ], [ 37, 38, 39 ], [ 40, 41, 42 ],
[ 43, 44, 45 ], 46, [ 47, 48, 49 ], 50, [ 51, 52, 53 ], 54, 55, 56, 57,
[ 58, 59, 60 ], [ 61, 62, 63 ], 64, [ 65, 66, 67 ], 68, [ 69, 70, 71 ],
[ 72, 73, 74 ], [ 75, 76, 77 ], [ 78, 79, 80 ], [ 81, 82, 83 ], 84, 85,
[ 86, 87, 88 ], [ 89, 90, 91 ], [ 92, 93, 94 ], 95, 96, [ 97, 98, 99 ],
[ 100, 101, 102 ], [ 103, 104, 105 ], [ 106, 107, 108 ], [ 109, 110, 111 ],
[ 112, 113, 114 ] ]
gap> ext:= CompositionMaps( ind[1], inv );
[ 57572775, 59535, 3591, 0, 0, 0, 0, 0, 2187, 0, 27, 135, 243, 0, 0, 0, 0, 0,
0, 0, 27, 0, 0, 27, 27, 0, 8, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ]
gap> perms:= PermChars( o8p33, rec( torso:= ext ) );
[ Character( CharacterTable( "08+(3).3" ), [ 57572775, 59535, 3591, 0, 0, 0,
0, 0, 2187, 0, 27, 135, 243, 0, 0, 0, 0, 0, 0, 27, 0, 27, 27, 0,
8, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 3159, 3159, 243, 243, 39, 39, 3, 3, 0, 0, 0, 0, 0, 0, 0, 0, 3, 3,
3, 3, 3, 3, 0, 0, 0, 0, 0, 0, 2, 2, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0
] ) ]
gap> PermCharInfo( o8p33, perms ).ATLAS;
[ "1a+780aabb+2457a+2808abc+9450aaabccc+18200abcdddef+24192a+54600a^5b+70200\
aabb+87360ab+139776a^5+147420a^4b^4+163800ab+184275aabc+199017aa+218700a\
+245700a+291200aef+332800a^4b^5c^5+491400aaabcd+531441a^5b^4c^4+55\
2825a^4+568620aabb+698880a^4b^4+716800aaabccddeeff+786240aabb+873600aa\
+998400aa+1257984a^6+1397760aa" ]

```

## 6 The Action of $O_7(3).2$ on the Cosets of $2^7.S_7$

We want to know whether the permutation character of  $O_7(3).2$  (see [CCN<sup>+</sup>85, p. 108]) on the cosets of its maximal subgroup  $U$  of type  $2^7.S_7$  is multiplicity-free.

As in the previous examples, first we try to compute the permutation character of the simple group  $O_7(3)$ . It turns out that the direct computation of all candidates from the degree is very time consuming. But we can use for example the additional information provided by the fact that  $U$  contains an  $A_7$  subgroup. We compute the possible class fusions.

```

gap> o73:= CharacterTable( "07(3)" );;
gap> a7:= CharacterTable( "A7" );;
gap> fus:= PossibleClassFusions( a7, o73 );
[ [ 1, 3, 6, 10, 15, 16, 24, 33, 33 ], [ 1, 3, 7, 10, 15, 16, 22, 33, 33 ] ]

```

We cannot decide easily which fusion is the right one, but already the fact that no other fusions are possible gives us some information about impossible constituents of the permutation character we want to compute.

```

gap> ind:= List( fus,
> x -> Induced( a7, o73, [ TrivialCharacter( a7 ) ], x )[1] );;
gap> mat:= MatScalarProducts( o73, Irr( o73 ), ind );;
gap> sum:= Sum( mat );
[ 2, 6, 2, 0, 8, 6, 2, 4, 4, 8, 3, 0, 4, 4, 9, 3, 5, 0, 0, 9, 0, 10, 5, 6,
15, 1, 12, 1, 15, 7, 2, 4, 14, 16, 0, 12, 12, 7, 8, 8, 14, 12, 12, 14, 6,
6, 20, 16, 12, 12, 12, 10, 10, 12, 12, 8, 12, 6 ]
gap> const:= Filtered( [ 1 .. Length( sum ) ], x -> sum[x] <> 0 );
[ 1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 13, 14, 15, 16, 17, 20, 22, 23, 24, 25, 26,

```

```

27, 28, 29, 30, 31, 32, 33, 34, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46,
47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58 ]
gap> Length( const );
52
gap> const:= Irr( o73 ){ const };;
gap> rat:= RationalizedMat( const );;
```

But much more can be deduced from the fact that certain zeros of the permutation character can be predicted.

```

gap> names:= ClassNames( o73 );
[ "1a", "2a", "2b", "2c", "3a", "3b", "3c", "3d", "3e", "3f", "3g", "4a",
  "4b", "4c", "4d", "5a", "6a", "6b", "6c", "6d", "6e", "6f", "6g", "6h",
  "6i", "6j", "6k", "6l", "6m", "6n", "6o", "6p", "7a", "8a", "8b", "9a",
  "9b", "9c", "9d", "10a", "10b", "12a", "12b", "12c", "12d", "12e", "12f",
  "12g", "12h", "13a", "13b", "14a", "15a", "18a", "18b", "18c", "18d", "20a"
]
gap> List( fus, x -> names{ x } );
[ [ "1a", "2b", "3b", "3f", "4d", "5a", "6h", "7a", "7a" ],
  [ "1a", "2b", "3c", "3f", "4d", "5a", "6f", "7a", "7a" ] ]
gap> torso:= [ 28431 ];;
gap> zeros:= [ 5, 8, 9, 11, 17, 20, 23, 28, 29, 32, 36, 37, 38,
>             43, 46, 47, 48, 53, 54, 55, 56, 57, 58 ];;
gap> names{ zeros };
[ "3a", "3d", "3e", "3g", "6a", "6d", "6g", "6l", "6m", "6p", "9a", "9b",
  "9c", "12b", "12e", "12f", "12g", "15a", "18a", "18b", "18c", "18d", "20a" ]
```

Every order 3 element of  $U$  lies in an  $A_7$  subgroup of  $U$ , so among the classes of element order 3, at most the classes 3B, 3C, and 3F can have nonzero permutation character values. The excluded classes of element order 6 are the square roots of the excluded order 3 elements, likewise the given classes of element orders 9, 12, and 18 are excluded. The character value on 20A must be zero because  $U$  does not contain elements of this order. So we enter the additional information about these zeros.

```

gap> for i in zeros do
>   torso[i]:= 0;
> od;
gap> torso;
[ 28431,,, 0,, 0, 0,, 0,,,,, 0,, 0,, 0,,,, 0, 0,, 0,,, 0, 0, 0,,,,, 0,
  ,, 0, 0, 0,,,,, 0, 0, 0, 0, 0, 0 ]
gap> perms:= PermChars( o73, rec( torso:= torso, chars:= rat ) );
[ Character( CharacterTable( "O7(3)" ), [ 28431, 567, 567, 111, 0, 0, 243, 0,
  0, 81, 0, 15, 3, 27, 15, 6, 0, 0, 27, 0, 3, 27, 0, 0, 0, 3, 9, 0, 0, 3,
  3, 0, 4, 1, 1, 0, 0, 0, 0, 2, 2, 3, 0, 3, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
  0, 0, 0, 0 ] ) ]
gap> PermCharInfo( o73, perms ).ATLAS;
[ "1a+78a+168a+182a+260ab+1092a+2457a+2730a+4095b+5460a+11648a" ]
```

We see that this character is already multiplicity free, so this holds also for its extension to  $O_7(3).2$ , and we need not compute this extension. (Of course we could compute it in the same way as in the examples above.)

## 7 The Action of $O_8^+(3).2_1$ on the Cosets of $2^7.A_8$

We are interested in the permutation character of  $O_8^+(3).2_1$  that corresponds to the action on the cosets of a subgroup of type  $2^7.A_8$ . The intersection of the point stabilizer with the simple group

$O_8^+(3)$  is of type  $2^6.A_8$ . First we compute the class fusion of these groups, modulo problems with ambiguities due to table automorphisms.

```

gap> o8p3:= CharacterTable( "08+(3)" );;
gap> o8p2:= CharacterTable( "08+(2)" );;
gap> fus:= PossibleClassFusions( o8p2, o8p3 );;
gap> NamesOfFusionSources( o8p2 );
[ "A9", "2^8:08+(2)", "08+(2)M2", "08+(2)M3", "08+(2)M5", "08+(2)M6",
  "08+(2)M8", "08+(2)M9", "(3xU4(2)):2", "08+(2)M11", "08+(2)M12",
  "2^(1+8)_+(S3xS3xS3)", "3^4:2^3.S4(a)", "(A5xA5):2^2", "08+(2)M16",
  "08+(2)M17", "2^(1+8)+.08+(2)", "2^6:A8", "2.08+(2)", "2^2.08+(2)", "S6(2)"
]
gap> sub:= CharacterTable( "2^6:A8" );;
gap> subfus:= GetFusionMap( sub, o8p2 );
[ 1, 3, 2, 2, 4, 5, 6, 13, 3, 6, 12, 13, 14, 7, 21, 24, 11, 30, 29, 31, 13,
  17, 15, 16, 14, 17, 36, 37, 18, 41, 24, 44, 48, 28, 33, 32, 34, 35, 35, 51,
  51 ]
gap> fus:= List( fus, x -> CompositionMaps( x, subfus ) );;
gap> fus:= Set( fus );;
gap> Length( fus );
24

```

The ambiguities due to Galois automorphisms disappear when we are looking for the permutation characters induced by the fusions.

```

gap> ind:= List( fus, x -> Induced( sub, o8p3,
  > [ TrivialCharacter( sub ) ], x )[1] );;
gap> ind:= Set( ind );;
gap> Length( ind );
6

```

Now we try to extend the candidates to  $O_8^+(3).2_1$ ; the choice of the fusion of  $O_8^+(3)$  into  $O_8^+(3).2_1$  determines which of the candidates may extend.

```

gap> o8p32:= CharacterTable( "08+(3).2_1" );;
gap> fus:= GetFusionMap( o8p3, o8p32 );;
gap> ext:= List( ind, x -> CompositionMaps( x, InverseMap( fus ) ) );;
gap> ext:= Filtered( ext, x -> ForAll( x, IsInt ) );
[ [ 3838185, 17577, 8505, 8505, 873, 0, 0, 0, 0, 6561, 0, 0, 729, 0, 9, 105,
  45, 45, 105, 30, 0, 0, 0, 0, 0, 0, 0, 0, 0, 189, 0, 0, 0, 9, 9, 27, 27,
  0, 0, 27, 9, 0, 8, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 2, 0, 0, 0, 0, 0,
  0, 0, 9, 0, 0, 0, 0, 0, 3, 0, 0, 0, 0, 0, 0, 0, 6, 0, 0, 0, 0, 0,
  0, 0 ], [ 3838185, 17577, 8505, 8505, 873, 0, 6561, 0, 0, 0, 0, 729,
  0, 9, 105, 45, 45, 105, 30, 0, 0, 0, 0, 189, 0, 0, 0, 9, 0, 0, 0,
  9, 27, 27, 0, 0, 9, 27, 0, 8, 1, 1, 0, 0, 0, 0, 0, 0, 2, 0, 0, 0,
  0, 0, 9, 0, 0, 0, 0, 0, 3, 0, 0, 0, 0, 0, 0, 0, 6, 0, 0, 0, 0, 0,
  0, 0, 0, 0, 0 ] ]

```

We compute the extensions of the first candidate; the other belongs to another class of subgroups, which is the image under an outer automorphism. (These calculations may need about one hour, depending on the computer used.)

```

gap> perms:= PermChars( o8p32, rec( torso:= ext[1] ) );
[ Character( CharacterTable( "08+(3).2_1" ),
  [ 3838185, 17577, 8505, 8505, 873, 0, 0, 0, 0, 6561, 0, 0, 729, 0, 9,

```

```

105, 45, 45, 105, 30, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 189, 0, 0, 0, 9, 9,
27, 27, 0, 0, 27, 9, 0, 8, 1, 1, 0, 0, 0, 0, 0, 0, 0, 2, 0, 0, 0, 0,
0, 0, 0, 0, 9, 0, 0, 0, 0, 0, 0, 3, 0, 0, 0, 0, 0, 0, 6, 0, 0, 0,
0, 0, 0, 0, 3159, 1575, 567, 63, 87, 15, 0, 0, 45, 0, 81, 9, 27, 0, 0,
3, 3, 3, 3, 5, 5, 0, 0, 0, 4, 0, 0, 27, 0, 9, 0, 0, 15, 0, 3, 0, 0, 2,
0, 0, 0, 0, 0, 3, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ] ) ]
gap> PermCharInfo( o8p32, perms ).ATLAS;
[ "1a+260abc+520ab+819a+2808b+9450aab+18200a+23400ac+29120b+36400aab+46592abce\
+49140d+66339a+98280ab+163800a+189540d+232960d+332800ab+368550a+419328a+531441\
ab" ]

```

Now we repeat the calculations for  $O_8^+(3).2_2$  instead of  $O_8^+(3).2_1$ .

```

gap> o8p32:= CharacterTable( "O8+(3).2_2" );;
gap> fus:= GetFusionMap( o8p3, o8p32 );;
gap> ext:= List( ind, x -> CompositionMaps( x, InverseMap( fus ) ) );;
gap> ext:= Filtered( ext, x -> ForAll( x, IsInt ) );;
gap> perms:= PermChars( o8p32, rec( torso:= ext[1] ) );
[ Character( CharacterTable( "O8+(3).2_2" ), [ 3838185, 17577, 8505, 873, 0,
0, 0, 6561, 0, 0, 0, 0, 729, 0, 9, 105, 45, 105, 30, 0, 0, 0, 0, 0, 0,
189, 0, 0, 0, 9, 0, 9, 27, 0, 0, 0, 27, 27, 9, 0, 8, 1, 1, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 2, 0, 0, 0, 0, 0, 9, 0, 0, 0, 0, 0, 0, 3, 0, 0, 0,
0, 0, 0, 0, 6, 0, 0, 0, 0, 0, 0, 199017, 2025, 297, 441, 73, 9, 0,
1215, 0, 0, 0, 0, 0, 81, 0, 0, 0, 0, 27, 27, 0, 1, 9, 12, 0, 0, 45, 0,
0, 1, 0, 0, 3, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2, 1, 0, 0, 0, 0, 0, 0
] ) ]
gap> PermCharInfo( o8p32, perms ).ATLAS;
[ "1a+260aac+520ab+819a+2808a+9450aaa+18200accee+23400ac+29120a+36400a+46592aa\
+49140c+66339a+93184a+98280ab+163800a+184275ac+189540c+232960c+332800aa+419328\
a+531441aa" ]

```

## 8 The Action of $S_4(4).4$ on the Cosets of $5^2.[2^5]$

We want to know whether the permutation character corresponding to the action of  $S_4(4).4$  (see [CCN<sup>+</sup>85, p. 44]) on the cosets of its maximal subgroup of type  $5^2 : [2^5]$  is multiplicity free.

The library names of subgroups for which the class fusions are stored are listed as value of the attribute `NamesOfFusionSources`, and for groups whose isomorphism type is not determined by the name this is the recommended way to find out whether the table of the subgroup is contained in the GAP library and known to belong to this group. (It might be that a table with such a name is contained in the library but belongs to another group, and it may also be that the table of the group is contained in the library –with any name– but it is not known that this group is isomorphic to a subgroup of  $S_4(4).4$ .)

```

gap> s444:= CharacterTable( "S4(4).4" );;
gap> NamesOfFusionSources( s444 );
[ "(L3(2)xS4(4):2).2", "S4(4)", "S4(4).2" ]

```

So we cannot simply fetch the table of the subgroup. As in the previous examples, we compute the possible permutation characters.

```

gap> perms:= PermChars( s444, rec( torso:= [ Size( s444 ) / ( 5^2*2^5 ) ] ) );
[ Character( CharacterTable( "S4(4).4" ), [ 4896, 384, 96, 0, 16, 32, 36, 16,
0, 4, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ] ),

```

```

Character( CharacterTable( "S4(4).4" ), [ 4896, 192, 32, 0, 0, 8, 6, 1, 0,
      2, 0, 0, 36, 0, 12, 0, 0, 0, 1, 0, 6, 6, 2, 2, 0, 0, 0, 0, 1, 1 ] ),
Character( CharacterTable( "S4(4).4" ), [ 4896, 240, 64, 0, 8, 8, 36, 16,
      0, 0, 0, 0, 0, 12, 8, 0, 4, 4, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ] ) ]

```

So there are three candidates. None of them is multiplicity free, so we need not decide which of the candidates actually belongs to the group  $5^2 : [2^5]$  we have in mind.

```

gap> PermCharInfo( s444, perms ).ATLAS;
[ "1abcd+50abcd+153abcd+170a{4}b{4}+680aabb",
  "1a+50ac+153a+170aab+256a+680abb+816a+1020a",
  "1ac+50ac+68a+153abcd+170aabbb+204a+680abb+1020a" ]

```

(If we would be interested which candidate is the right one, we could for example look at the intersection with  $S_4(4)$ , and hope for a contradiction to the fact that the group must lie in a  $(A_5 \times A_5) : 2$  subgroup.)

## 9 The Action of $C_{O_1}$ on the Cosets of Involution Centralizers

We compute the permutation characters of the sporadic simple Conway group  $C_{O_1}$  (see [CCN<sup>+</sup>85, p. 180]) corresponding to the actions on the cosets of involution centralizers. Equivalently, we are interested in the action of  $C_{O_1}$  on conjugacy classes of involutions. These characters can be computed as follows. First we take the table of  $C_{O_1}$ .

```

gap> t:= CharacterTable( "Co1" );
CharacterTable( "Co1" )

```

The centralizer of each 2A element is a maximal subgroup of  $C_{O_1}$ . This group is also contained in the table library. So we can compute the permutation character by explicit induction, and the decomposition in irreducibles is computed with the command `PermCharInfo`.

```

gap> s:= CharacterTable( Maxes( t )[5] );
CharacterTable( "2{1+8}.08+(2)" )
gap> ind:= Induced( s, t, [ TrivialCharacter( s ) ] );
gap> PermCharInfo( t, ind ).ATLAS;
[ "1a+299a+17250a+27300a+80730a+313950a+644644a+2816856a+5494125a+12432420a+24\
794000a" ]

```

The centralizer of a 2B element is not maximal. First we compute which maximal subgroup can contain it. The character tables of all maximal subgroups of  $C_{O_1}$  are contained in the GAP's table library, so we may take these tables and look at the group orders.

```

gap> centorder:= SizesCentralizers( t )[3];
gap> maxes:= List( Maxes( t ), CharacterTable );
gap> cand:= Filtered( maxes, x -> Size( x ) mod centorder = 0 );
[ CharacterTable( "(A4xG2(4)):2" ) ]
gap> u:= cand[1];
gap> index:= Size( u ) / centorder;
3

```

So there is a unique class of maximal subgroups containing the centralizer of a 2B element, as a subgroup of index 3. We compute the unique permutation character of degree 3 of this group, and induce this character to  $G$ .



```

true
gap> ind:= Induced( u, t, subperm );
[ Character( CharacterTable( "Co1" ), [ 10680579000, 1988280, 196560, 94744,
    0, 17010, 0, 945, 7560, 3432, 2280, 1728, 252, 308, 0, 225, 0, 0, 0,
    270, 0, 306, 0, 46, 45, 25, 0, 0, 120, 32, 12, 52, 36, 36, 0, 0, 0, 0,
    0, 45, 15, 0, 9, 3, 0, 0, 0, 0, 18, 0, 30, 0, 6, 18, 0, 3, 5, 0, 0, 0,
    0, 0, 0, 0, 0, 2, 2, 0, 0, 0, 0, 3, 0, 0, 0, 0, 1, 0, 0, 0, 0, 6, 0, 2,
    0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ] ) ]
gap> PermCharInfo( t, ind ).ATLAS;
[ "1a+17250aa+27300a+80730aa+644644aaa+871884a+1821600a+2055625aaa+2816856a+54\
94125a^4+12432420aa+16347825aa+23244375a+24174150aa+24667500aa+24794000aaa+3\
1574400a+40370176a+55255200a+66602250a^4+83720000aa+85250880aaa+91547820aa+1\
06142400a+150732800a+184184000aaa+185912496aaa+185955000aaa+207491625aaa+21554\
7904aa+241741500aaa+247235625a+257857600aa+259008750a+280280000a+302176875a+32\
6956500a+387317700a+402902500a+464257024a+469945476b+502078500a+503513010a+504\
627200a+522161640a" ]

```

## 10 The Multiplicity Free Permutation Characters of $G_2(3)$

We compute the multiplicity free possible permutation characters of  $G_2(3)$  (see [CCN<sup>+</sup>85, p. 60]).

For each divisor  $d$  of the group order, we compute all those possible permutation characters of degree  $d$  of  $G$  for which each irreducible constituent occurs with multiplicity at most 1; this is done by prescribing the `maxmult` component of the second argument of `PermChars` to be the list with 1 at each position.

```

gap> t:= CharacterTable( "G2(3)" );
CharacterTable( "G2(3)" )
gap> t:= CharacterTable( "G2(3)" );;
gap> n:= Length( RationalizedMat( Irr( t ) ) );;
gap> maxmult:= List( [ 1 .. n ], i -> 1 );;
gap> perms:= [];;
gap> divs:= DivisorsInt( Size( t ) );;
gap> for d in divs do
>   Append( perms,
>     PermChars( t, rec( bounds := false,
>                       degree := d,
>                       maxmult := maxmult ) ) );;
>   od;
gap> Length( perms );
42
gap> List( perms, Degree );
[ 1, 351, 351, 364, 364, 378, 378, 546, 546, 546, 546, 546, 702, 702, 728,
  728, 1092, 1092, 1092, 1092, 1092, 1092, 1092, 1092, 1456, 1456, 1638,
  1638, 2184, 2184, 2457, 2457, 2457, 2457, 3159, 3276, 3276, 3276,
  4368, 6552, 6552 ]

```

For finding out which of these candidates are really permutation characters, we could inspect them piece by piece, using the information in [CCN<sup>+</sup>85]. For example, the candidates of degrees 351, 364, and 378 are induced from the trivial characters of maximal subgroups of  $G$ , whereas the candidates of degree 546 are not permutation characters.

Since the table of marks of  $G$  is available in GAP, we can extract all permutation characters from the table of marks, and then filter out the multiplicity free ones.

```

gap> tom:= TableOfMarks( "G2(3)" );
TableOfMarks( "G2(3)" )
gap> tbl:= CharacterTable( "G2(3)" );
CharacterTable( "G2(3)" )
gap> permstom:= PermCharsTom( tbl, tom );;
gap> Length( permstom );
433
gap> multfree:= Intersection( perms, permstom );;
gap> Length( multfree );
15
gap> List( multfree, Degree );
[ 1, 351, 351, 364, 364, 378, 378, 702, 702, 728, 728, 1092, 1092, 2184, 2184
]

```

## 11 Degree 11 200 Permutation Characters of $O_8^+(2)$

We compute the primitive permutation characters of degree 11 200 of  $O_8^+(2)$  and  $O_8^+(2).2$  (see [CCN<sup>+</sup>85, p. 85]). The character table of the maximal subgroup of type  $3^4 : 2^3.S_4$  in  $O_8^+(2)$  is not available in the GAP table library. But the group extends to a wreath product of  $S_3$  and  $S_4$  in the group  $O_8^+(2).2$ , and the table of this wreath product can be constructed easily.

```

gap> tbl2:= CharacterTable("08+(2).2");;
gap> s3:= CharacterTable( "Symmetric", 3 );;
gap> s:= CharacterTableWreathSymmetric( s3, 4 );
CharacterTable( "Sym(3)wrS4" )

```

The permutation character  $\pi$  of  $O_8^+(2).2$  can thus be computed by explicit induction, and the character of  $O_8^+(2)$  is obtained by restriction of  $\pi$ .

```

gap> fus:= PossibleClassFusions( s, tbl2 );
[ [ 1, 41, 6, 3, 48, 9, 42, 19, 51, 8, 5, 50, 24, 49, 7, 2, 44, 22, 42, 12,
    53, 17, 58, 21, 5, 47, 26, 50, 37, 52, 23, 60, 18, 4, 46, 25, 14, 61,
    20, 9, 53, 30, 51, 26, 64, 8, 52, 31, 13, 56, 38 ] ]
gap> pi:= Induced( s, tbl2, [ TrivialCharacter( s ) ], fus[1] )[1];
Character( CharacterTable( "08+(2).2" ), [ 11200, 256, 160, 160, 80, 40, 40,
    76, 13, 0, 0, 8, 8, 4, 0, 0, 16, 16, 4, 4, 4, 1, 1, 1, 1, 5, 0, 0, 0, 1, 1,
    0, 0, 0, 0, 0, 2, 2, 0, 0, 1120, 96, 0, 16, 0, 16, 8, 10, 4, 6, 7, 12, 3,
    0, 0, 2, 0, 4, 0, 1, 1, 0, 0, 1, 0, 0, 0 ] )
gap> PermCharInfo( tbl2, pi ).ATLAS;
[ "1a+84a+168a+175a+300a+700c+972a+1400a+3200a+4200b" ]
gap> tbl:= CharacterTable( "08+(2)" );
CharacterTable( "08+(2)" )
gap> rest:= RestrictedClassFunction( pi, tbl );
Character( CharacterTable( "08+(2)" ), [ 11200, 256, 160, 160, 160, 80, 40,
    40, 40, 76, 13, 0, 0, 8, 8, 8, 4, 0, 0, 0, 16, 16, 16, 4, 4, 4, 4, 1, 1, 1,
    1, 1, 1, 5, 0, 0, 0, 1, 1, 1, 0, 0, 0, 0, 0, 0, 2, 2, 2, 0, 0, 0 ] )
gap> PermCharInfo( tbl, rest ).ATLAS;
[ "1a+84abc+175a+300a+700bcd+972a+3200a+4200a" ]

```

## 12 A Proof of Nonexistence of a Certain Subgroup

We prove that the sporadic simple Mathieu group  $G = M_{22}$  (see [CCN<sup>+</sup>85, p. 39]) has no subgroup of index 56. In [Isa76], remark after Theorem 5.18, this is stated as an example of the case that a character may be a possible permutation character but not a permutation character.

Let us consider the possible permutation character of degree 56 of  $G$ .

```

gap> tbl:= CharacterTable( "M22" );
CharacterTable( "M22" )
gap> perms:= PermChars( tbl, rec( torso:= [ 56 ] ) );
[ Character( CharacterTable( "M22" ), [ 56, 8, 2, 4, 0, 1, 2, 0, 0, 2, 1, 1
] ) ]
gap> pi:= perms[1];;
gap> Norm( pi );
2
gap> Display( tbl, rec( chars:= perms ) );
M22

      2 7 7 2 5 4 . 2 . . 3 . .
      3 2 1 2 . . . 1 . . . . .
      5 1 . . . . 1 . . . . .
      7 1 . . . . . 1 1 . . .
     11 1 . . . . . . . . 1 1

      1a 2a 3a 4a 4b 5a 6a 7a 7b 8a 11a 11b
     2P 1a 1a 3a 2a 2a 5a 3a 7a 7b 4a 11b 11a
     3P 1a 2a 1a 4a 4b 5a 2a 7b 7a 8a 11a 11b
     5P 1a 2a 3a 4a 4b 1a 6a 7b 7a 8a 11a 11b
     7P 1a 2a 3a 4a 4b 5a 6a 1a 1a 8a 11b 11a
    11P 1a 2a 3a 4a 4b 5a 6a 7a 7b 8a 1a 1a

     Y.1 56 8 2 4 . 1 2 . . 2 1 1

```

Suppose that  $\pi$  is a permutation character of  $G$ . Since  $G$  is 2-transitive on the 56 cosets of the point stabilizer  $S$ , this stabilizer is transitive on 55 points, and thus  $G$  has a subgroup  $U$  of index  $56 \cdot 55 = 3080$ . We compute the possible permutation character of this degree.

```

gap> perms:= PermChars( tbl, rec( torso:= [ 56 * 55 ] ) );;
gap> Length( perms );
16

```

$U$  is contained in  $S$ , so only those candidates must be considered that vanish on all classes where  $\pi$  vanishes. Furthermore, the index of  $U$  in  $S$  is odd, so the Sylow 2 subgroups of  $U$  and  $S$  are isomorphic;  $S$  contains elements of order 8, hence also  $U$  does.

```

gap> OrdersClassRepresentatives( tbl );
[ 1, 2, 3, 4, 4, 5, 6, 7, 7, 8, 11, 11 ]
gap> perms:= Filtered( perms, x -> x[5] = 0 and x[10] <> 0 );
[ Character( CharacterTable( "M22" ), [ 3080, 56, 2, 12, 0, 0, 2, 0, 0, 2, 0,
0 ] ), Character( CharacterTable( "M22" ),
[ 3080, 8, 2, 8, 0, 0, 2, 0, 0, 4, 0, 0 ] ),
Character( CharacterTable( "M22" ), [ 3080, 24, 11, 4, 0, 0, 3, 0, 0, 2, 0,
0 ] ), Character( CharacterTable( "M22" ),
[ 3080, 24, 20, 4, 0, 0, 0, 0, 0, 2, 0, 0 ] ) ]

```

For getting an overview of the distribution of the elements of  $U$  to the conjugacy classes of  $G$ , we use the output of `PermCharInfo`.

```
gap> infoperms:= PermCharInfo( tbl, perms );
gap> Display( tbl, infoperms.display );
M22
```

```

  2   7  7  2  5  2  3
  3   2  1  2  .  1  .
  5   1  .  .  .  .  .
  7   1  .  .  .  .  .
 11   1  .  .  .  .  .
```

```

          1a 2a 3a 4a 6a 8a
2P   1a 1a 3a 2a 3a 4a
3P   1a 2a 1a 4a 2a 8a
5P   1a 2a 3a 4a 6a 8a
7P   1a 2a 3a 4a 6a 8a
11P  1a 2a 3a 4a 6a 8a
```

```

I.1   3080 56  2 12  2  2
I.2     1 21  8 54 24 36
I.3     1  3  4  9 12 18
I.4   3080  8  2  8  2  4
I.5     1  3  8 36 24 72
I.6     1  3  4  9 12 18
I.7   3080 24 11  4  3  2
I.8     1  9 44 18 36 36
I.9     1  3  4  9 12 18
I.10  3080 24 20  4  .  2
I.11     1  9 80 18  . 36
I.12     1  3  4  9 12 18
```

We have four candidates. For each the above list shows first the character values, then the cardinality of the intersection of  $U$  with the classes, and then lower bounds for the lengths of  $U$ -conjugacy classes of these elements. Only those classes of  $G$  are shown that contain elements of  $U$  for at least one of the characters.

If the first two candidates are permutation characters corresponding to  $U$  then  $U$  contains exactly 8 elements of order 3 and thus  $U$  has a normal Sylow 3 subgroup  $P$ . But the order of  $N_G(P)$  is bounded by 72, which can be shown as follows. The only elements in  $G$  with centralizer order divisible by 9 are of order 1 or 3, so  $P$  is self-centralizing in  $G$ . The factor  $N_G(P)/C_G(P)$  is isomorphic with a subgroup of  $\text{Aut}(G) \cong GL(2,3)$  which has order divisible by 16, hence the order of  $N_G(P)$  divides 144. Now note that  $[G : N_G(P)] \equiv 1 \pmod{3}$  by Sylow's Theorem, and  $|G|/144 = 3080 \equiv -1 \pmod{3}$ . Thus the first two candidates are not permutation characters.

If the last two candidates are permutation characters corresponding to  $U$  then  $U$  has self-normalizing Sylow subgroups. This is because the index of a Sylow 2 normalizer in  $G$  is odd and divides 9, and if it is smaller than 9 then  $U$  contains at most  $3 \cdot 15 + 1$  elements of 2 power order; the index of a Sylow 3 normalizer in  $G$  is congruent to 1 modulo 3 and divides 16, and if it is smaller than 16 then  $U$  contains at most  $4 \cdot 8$  elements of order 3.

But since  $U$  is solvable and not a  $p$ -group, not all its Sylow subgroups can be self-normalizing; note that  $U$  has a proper normal subgroup  $N$  containing a Sylow  $p$  subgroup  $P$  of  $U$  for a prime divisor  $p$  of  $|U|$ , and  $U = N \cdot N_U(P)$  holds by the Frattini argument (see [Hup67, Satz I.7.8]).

## 13 A Permutation Character of the Lyons group

Let  $G$  be a maximal subgroup with structure  $3^{2+4} : 2A_5.D_8$  in the sporadic simple Lyons group  $Ly$ . We want to compute the permutation character  $1_G^{Ly}$ . (This construction has been explained in [BP98, Section 4.2], without showing explicit GAP code.)

In the representation of  $Ly$  as automorphism group of the rank 5 graph  $B$  with 9606125 points (see [CCN<sup>+</sup>85, p. 174]),  $G$  is the stabilizer of an edge. A group  $S$  with structure  $3.McL.2$  is the point stabilizer. So the two point stabilizer  $U = S \cap G$  is a subgroup of index 2 in  $G$ . The index of  $U$  in  $S$  is 15400, and according to the list of maximal subgroups of  $McL.2$  (see [CCN<sup>+</sup>85, p. 100]), the group  $U$  is isomorphic to the preimage in  $3.McL.2$  of a subgroup  $H$  of  $McL.2$  with structure  $3_+^{1+4} : 4S_5$ .

Using the improved combinatorial method described in [BP98, Section 3.2], all possible permutation characters of degree 15400 for the group  $McL$  are computed. (The method of [BP98, Section 3.3] is slower but also needs only a few seconds.)

```
gap> ly:= CharacterTable( "Ly" );;
gap> mcl:= CharacterTable( "McL" );;
gap> mcl2:= CharacterTable( "McL.2" );;
gap> 3mcl2:= CharacterTable( "3.McL.2" );;
gap> perms:= PermChars( mcl, rec( degree:= 15400 ) );
[ Character( CharacterTable( "McL" ), [ 15400, 56, 91, 10, 12, 25, 0, 11, 2,
  0, 0, 2, 1, 1, 1, 0, 0, 3, 0, 0, 1, 1, 1, 1 ] ),
  Character( CharacterTable( "McL" ), [ 15400, 280, 10, 37, 20, 0, 5, 10, 1,
  0, 0, 2, 1, 1, 0, 0, 0, 2, 0, 0, 0, 0, 0, 0 ] ) ]
```

We get two characters, corresponding to the two classes of maximal subgroups of index 15400 in  $McL$ . The permutation character  $\pi = 1_{H \cap McL}^{McL}$  is the one with nonzero value on the class 10A, since the subgroup of structure  $2S_5$  in  $H \cap McL$  contains elements of order 10.

```
gap> ord10:= Filtered( [ 1 .. NrConjugacyClasses( mcl ) ],
> i -> OrdersClassRepresentatives( mcl )[i] = 10 );
[ 15 ]
gap> List( perms, pi -> pi[ ord10[1] ] );
[ 1, 0 ]
gap> pi:= perms[1];
Character( CharacterTable( "McL" ), [ 15400, 56, 91, 10, 12, 25, 0, 11, 2, 0,
  0, 2, 1, 1, 1, 0, 0, 3, 0, 0, 1, 1, 1, 1 ] )
```

The character  $1_H^{McL.2}$  is an extension of  $\pi$ , so we can use the method of [BP98, Section 3.3] to compute all possible permutation characters for the group  $McL.2$  that have the values of  $\pi$  on the classes of  $McL$ . We find that the extension of  $\pi$  to a permutation character of  $McL.2$  is unique. Regarded as a character of  $3.McL.2$ , this character is equal to  $1_U^S$ .

```
gap> map:= InverseMap( GetFusionMap( mcl, mcl2 ) );
[ 1, 2, 3, 4, 5, 6, 7, 8, 9, [ 10, 11 ], 12, [ 13, 14 ], 15, 16, 17, 18,
  [ 19, 20 ], [ 21, 22 ], [ 23, 24 ] ]
gap> torso:= CompositionMaps( pi, map );
[ 15400, 56, 91, 10, 12, 25, 0, 11, 2, 0, 2, 1, 1, 0, 0, 3, 0, 1, 1 ]
gap> perms:= PermChars( mcl2, rec( torso:= torso ) );
[ Character( CharacterTable( "McL.2" ), [ 15400, 56, 91, 10, 12, 25, 0, 11,
  2, 0, 2, 1, 1, 0, 0, 3, 0, 1, 1, 110, 26, 2, 4, 0, 0, 5, 2, 1, 1, 0, 0,
  1, 1 ] ) ]
gap> pi:= Inflated( perms[1], 3mcl2 );
Character( CharacterTable( "3.McL.2" ), [ 15400, 15400, 56, 56, 91, 91, 10,
  12, 12, 25, 25, 0, 0, 11, 11, 2, 2, 0, 0, 0, 2, 2, 1, 1, 1, 0, 0, 0, 3,
  3, 0, 0, 0, 1, 1, 1, 1, 1, 1, 110, 26, 2, 4, 0, 0, 5, 2, 1, 1, 0, 0, 1, 1
  ] )
```

The fusion of conjugacy classes of  $S$  in  $Ly$  can be computed from the character tables of  $S$  and  $Ly$  given in [CCN<sup>+</sup>85], it is unique up to Galois automorphisms of the table of  $Ly$ .

```
gap> fus:= PossibleClassFusions( 3mcl2, ly );; Length( fus );
4
gap> g:= AutomorphismsOfTable( ly );;
gap> OrbitLengths( g, fus, OnTuples );
[ 4 ]
```

Now we can induce  $1_U^S$  to  $Ly$ , which yields  $(1_U^S)^{Ly} = 1_U^{Ly}$ .

```
gap> pi:= Induced( 3mcl2, ly, [ pi ], fus[1] )[1];
Character( CharacterTable( "Ly" ), [ 147934325000, 286440, 1416800, 1082,
784, 12500, 0, 672, 42, 24, 0, 40, 0, 2, 20, 0, 0, 0, 64, 10, 0, 50, 2, 0,
0, 4, 0, 0, 0, 0, 4, 0, 0, 0, 0, 2, 2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0 ] )
```

All elements of odd order in  $G$  are contained in  $U$ , for such an element  $g$  we have

$$1_G^{Ly}(g) = \frac{|C_{Ly}(g)|}{|G|} \cdot |G \cap Cl_{Ly}(g)| = \frac{|C_{Ly}(g)|}{2 \cdot |U|} \cdot |U \cap Cl_{Ly}(g)| = \frac{1}{2} \cdot 1_U^{Ly}(g),$$

so we can prescribe the values of  $1_G^{Ly}$  on all classes of odd element order. For elements  $g$  of even order we have the weaker condition  $U \cap Cl_{Ly}(g) \subseteq G \cap Cl_{Ly}(g)$  and thus  $1_G^{Ly}(g) \geq \frac{1}{2} \cdot 1_U^{Ly}(g)$ , which gives lower bounds for the value of  $1_G^{Ly}$  on the remaining classes.

```
gap> orders:= OrdersClassRepresentatives( ly );
[ 1, 2, 3, 3, 4, 5, 5, 6, 6, 6, 7, 8, 8, 9, 10, 10, 11, 11, 12, 12, 14, 15,
15, 15, 18, 20, 21, 21, 22, 22, 24, 24, 24, 25, 28, 30, 30, 31, 31, 31, 31,
31, 33, 33, 37, 37, 40, 40, 42, 42, 67, 67, 67 ]
gap> torso:= [];
gap> for i in [ 1 .. Length( orders ) ] do
>   if orders[i] mod 2 = 1 then
>     torso[i]:= pi[i]/2;
>   fi;
> od;
gap> torso;
[ 73967162500,, 708400, 541,, 6250, 0,,, 0,,, 1,,, 0, 0,,, 25, 1, 0,,, 0, 0,
,,,,, 0,,, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,,,,, 0, 0, 0 ]
```

Exactly one possible permutation character of  $Ly$  satisfies these conditions.

```
gap> perms:= PermChars( ly, rec( torso:= torso ) );;
gap> Length( perms );
43
gap> perms:= Filtered( perms, cand -> ForAll( [ 1 .. Length( orders ) ],
>   i -> cand[i] >= pi[i] / 2 ) );
[ Character( CharacterTable( "Ly" ), [ 73967162500, 204820, 708400, 541, 392,
6250, 0, 1456, 61, 25, 0, 22, 10, 1, 10, 0, 0, 0, 32, 5, 0, 25, 1, 0,
1, 2, 0, 0, 0, 0, 4, 1, 1, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2, 2,
0, 0, 0, 0, 0 ] ) ]
```

(The permutation character  $1_G^{Ly}$  was used in the proof that the character  $\chi_{37}$  of  $Ly$  (see [CCN<sup>+</sup>85, p. 175]) occurs with multiplicity at least 2 in each character of  $Ly$  that is induced from a proper subgroup of  $Ly$ .)

## 14 Identifying two subgroups of $\text{Aut}(U_3(5))$ (October 2001)

According to the Atlas of Finite Groups [CCN<sup>+</sup>85, p. 34], the group  $\text{Aut}(U_3(5))$  has two classes of maximal subgroups of order  $2^4 \cdot 3^3$ , which have the structures  $3^2:2S_4$  and  $6^2:D_{12}$ , respectively.

```
gap> tbl:= CharacterTable( "U3(5).3.2" );
CharacterTable( "U3(5).3.2" )
gap> deg:= Size( tbl ) / ( 2^4*3^3 );
1750
gap> pi:= PermChars( tbl, rec( torso:= [ deg ] ) );
[ Character( CharacterTable( "U3(5).3.2" ), [ 1750, 70, 13, 2, 0, 0, 1, 0, 0,
0, 10, 7, 10, 4, 2, 0, 0, 0, 0, 0, 0, 30, 10, 3, 0, 0, 1, 0, 0 ] ),
Character( CharacterTable( "U3(5).3.2" ), [ 1750, 30, 4, 6, 0, 0, 0, 0, 0,
0, 40, 7, 0, 6, 0, 0, 0, 0, 0, 0, 0, 20, 0, 2, 2, 0, 0, 0, 0 ] ) ]
```

Now the question is which character belongs to which subgroup. We see that the first character vanishes on the classes of element order 8 and the second does not, so only the first one can be the permutation character induced from  $6^2:D_{12}$ .

```
gap> ord8:= Filtered( [ 1 .. NrConjugacyClasses( tbl ) ],
> i -> OrdersClassRepresentatives( tbl )[i] = 8 );
[ 9, 25 ]
gap> List( pi, x -> x{ ord8 } );
[ [ 0, 0 ], [ 0, 2 ] ]
```

Thus the question is whether the second candidate is really a permutation character. Since none of the two candidates vanishes on any outer coset of  $U_3(5)$  in  $\text{Aut}(U_3(5))$ , the point stabilizers are extensions of groups of order  $2^3 \cdot 3^2$  in  $U_3(5)$ . The restrictions of the candidates to  $U_3(5)$  are different, so we can try to answer the question using information about this group.

```
gap> subtbl:= CharacterTable( "U3(5)" );
CharacterTable( "U3(5)" )
gap> rest:= RestrictedClassFunctions( pi, subtbl );
[ Character( CharacterTable( "U3(5)" ), [ 1750, 70, 13, 2, 0, 0, 0, 0, 1, 0,
0, 0, 0, 0 ] ), Character( CharacterTable( "U3(5)" ),
[ 1750, 30, 4, 6, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ] ) ]
```

The intersection of the  $3^2:2S_4$  subgroup with  $U_3(5)$  lies inside the maximal subgroup of type  $M_{10}$ , which does not contain elements of order 6. Only the second character has this property.

```
gap> ord6:= Filtered( [ 1 .. NrConjugacyClasses( subtbl ) ],
> i -> OrdersClassRepresentatives( subtbl )[i] = 6 );
[ 9 ]
gap> List( rest, x -> x{ ord6 } );
[ [ 1 ], [ 0 ] ]
```

In order to establish the two characters as permutation characters, we could also compute the permutation characters of the degree in question directly from the table of marks of  $U_3(5)$ , which is contained in the GAP library of tables of marks.

```
gap> tom:= TableOfMarks( "U3(5)" );
TableOfMarks( "U3(5)" )
gap> perms:= PermCharsTom( subtbl, tom );
gap> Set( Filtered( perms, x -> x[1] = deg ) ) = Set( rest );
true
```

We were mainly interested in the multiplicities of irreducible characters in these characters. The action of  $\text{Aut}(U_3(5))$  on the cosets of  $3^2:2S_4$  turns out to be multiplicity-free whereas that on the cosets of  $6^2:D_{12}$  is not.

```
gap> PermCharInfo( tbl, pi ).ATLAS;
[ "1a+21a+42a+84aac+105a+125a+126a+250a+252a+288bc",
  "1a+42a+84ac+105ab+125a+126a+250a+252b+288bc" ]
```

It should be noted that the restrictions of the multiplicity-free character to the subgroups  $U_3(5).2$  and  $U_3(5).3$  of  $\text{Aut}(U_3(5))$  are not multiplicity-free.

```
gap> subtbl2:= CharacterTable( "U3(5).2" );;
gap> rest2:= RestrictedClassFunctions( pi, subtbl2 );;
gap> PermCharInfo( subtbl2, rest2 ).ATLAS;
[ "1a+21aab+28aa+56aa+84a+105a+125aab+126aab+288aa",
  "1a+21ab+28a+56a+84a+105ab+125aab+126a+252a+288aa" ]
gap> subtbl3:= CharacterTable( "U3(5).3" );;
gap> rest3:= RestrictedClassFunctions( pi, subtbl3 );;
gap> PermCharInfo( subtbl3, rest3 ).ATLAS;
[ "1a+21abc+84aab+105a+125abc+126abc+144bcef",
  "1a+21bc+84ab+105aa+125abc+126adg+144bcef" ]
```

## 15 A Permutation Character of $\text{Aut}(O_8^+(2))$ (October 2001)

According to the Atlas of Finite Groups [CCN<sup>+</sup>85, p. 85], the group  $G = \text{Aut}(O_8^+(2))$  has a class of maximal subgroups of order  $2^{13} \cdot 3^2$ , thus the index of these subgroups in  $G$  is  $3^4 \cdot 5^2 \cdot 7$ . The intersection of these subgroups with  $H = O_8^+(2)$  lie inside maximal subgroups of type  $2^6:A_8$ . We want to show that the permutation character of the action of  $G$  on the cosets of these subgroups is not multiplicity-free.

Since the table of marks for  $H$  is available in GAP, but not that for  $G$ , we first compute the  $H$ -permutation characters of the intersections with  $H$  of index  $3^4 \cdot 5^2 \cdot 7 = 14175$  subgroups in  $G$ .

(Note that these intersections have order  $2^{12} \cdot 3$  because subgroups of order  $2^{12} \cdot 3^2$  are contained in  $O_8^+(2).2$  and hence are not maximal in  $G$ .)

```
gap> t:= CharacterTable( "08+(2).3.2" );;
gap> s:= CharacterTable( "08+(2)" );;
gap> tom:= TableOfMarks( s );;
gap> perms:= PermCharsTom( s, tom );;
gap> deg:= 3^4*5^2*7;
14175
gap> perms:= Filtered( perms, x -> x[1] = deg );;
gap> Length( perms );
4
gap> Length( Set( perms ) );
1
```

We see that there are four classes of subgroups  $S$  in  $H$  that may belong to maximal subgroups of the desired index in  $G$ , and that the permutation characters are equal. They lead to such groups if they extend to  $G$ , so we compute the possible permutation characters of  $G$  that extend these characters.

```
gap> fus:= PossibleClassFusions( s, t );
[ [ 1, 2, 3, 3, 3, 4, 5, 5, 6, 7, 8, 9, 10, 10, 10, 11, 12, 12, 12, 13,
    13, 13, 14, 14, 14, 15, 16, 16, 17, 17, 17, 18, 19, 20, 21, 22, 22,
```

```

22, 23, 23, 23, 24, 24, 24, 25, 26, 26, 26, 27, 27, 27 ] ]
gap> fus:= fus[1];;
gap> inv:= InverseMap( fus );;
gap> comp:= CompositionMaps( perms[1], inv );
[ 14175, 1215, 375, 79, 0, 0, 27, 27, 99, 15, 7, 0, 0, 0, 0, 9, 3, 1, 0, 1,
  1, 0, 0, 0, 0, 0, 0 ]
gap> ext:= PermChars( t, rec( torso:= comp ) );
[ Character( CharacterTable( "08+(2).3.2" ),
  [ 14175, 1215, 375, 79, 0, 0, 27, 27, 99, 15, 7, 0, 0, 0, 0, 9, 3, 1, 0,
    1, 1, 0, 0, 0, 0, 0, 0, 63, 9, 15, 7, 1, 0, 3, 3, 3, 1, 0, 0, 1, 1,
    945, 129, 45, 69, 21, 25, 13, 0, 0, 0, 9, 0, 3, 3, 7, 1, 0, 0, 0, 3, 1,
    0, 0, 0, 0, 0, 0 ] ) ]
gap> PermCharInfo( t, ext[1] ).ATLAS;
[ "1a+50b+100a+252bb+300b+700b+972bb+1400a+1944a+3200b+4032b" ]

```

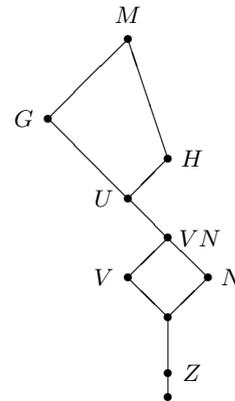
Thus we get one permutation character of  $G$  which is not multiplicity-free.

## 16 Four Primitive Permutation Characters of the Monster Group

In this section, we compute four primitive permutation characters  $1_H^M$  of the sporadic simple Monster group  $M$ , using the following strategy.

Let  $E$  be an elementary abelian 2-subgroup of  $M$ , and  $H = N_M(E)$ . For an involution  $z \in E$ , let  $G = C_M(z)$  and  $U = G \cap H = C_H(z)$  and  $V = C_H(E)$ , a normal subgroup of  $H$ . According to the Atlas of Finite Groups [CCN<sup>+</sup>85, p. 234],  $G$  has the structure  $2.B$  if  $z$  is in the class 2A of  $M$ , and  $G$  has the structure  $2_+^{1+24}.Co_1$  if  $z$  is in the class 2B of  $M$ . In the latter case, let  $N$  denote the extraspecial normal subgroup of order  $2^{25}$  in  $G$ . It will turn out that in our situation,  $U$  contains  $N$ .

We want to compute many values of  $1_H^M$  from the knowledge of permutation characters  $1_X^M$ , for suitable subgroups  $X$  with the property  $V \leq X \leq U$ , and then use the GAP function `PermChars` for computing all those possible permutation characters of  $M$  that take the known values; if there is a unique solution then this is the desired character  $1_H^M$ .



Why does this approach have a chance to be successful? Currently we do not have representations for the subgroups  $H$  in question, but the character tables of the involution centralizers  $G$  in  $M$  are available, and also either the character tables of  $X/V$  for the interesting subgroups  $X$  are known or we have enough information to compute the characters  $1_X^G$ .

And how do we compute certain values of  $1_H^M$ ? Suppose that  $\mathcal{C}$  is a union of classes of  $M$  and  $I$  is an index set such that  $(1_H)c_{\mathcal{C}} = (\sum_{i \in I} c_i 1_{X_i}^H)c_{\mathcal{C} \cap H}$  holds for suitable rational numbers  $c_i$ .

The right hand side of this equality lives in  $H/V$ , provided that  $\mathcal{C}$  “behaves well” w.r.t. factoring out the normal subgroup  $V$  of  $H$ , i. e., if there is a set of classes in  $H/V$  whose preimages in  $H$  form the set  $H \cap \mathcal{C}$ . For example,  $\mathcal{C}$  may be the set of all those elements in  $M$  whose order is not divisible by a particular prime  $p$  that divides  $|H|$  but not  $|U|$ .

Under these conditions, we have  $(1_H^M)c = ((\sum_{i \in I} c_i 1_{X_i}^G)^M)c$ , and we interpret the right hand side as follows: If  $X_i$  contains  $N$  then  $1_{X_i}^G$  can be identified with  $1_{X_i/N}^{G/N}$ . If  $X_i$  contains at least  $Z$  then

$1_{X_i}^G$  can be identified with  $1_{X_i/Z}^{G/Z}$ . As mentioned above, we have good chances to compute these characters. So the main task in each of the following sections is to find, for a suitable set  $\mathcal{C}$  of classes, a linear combination of permutation characters of  $H/V$  whose restriction to  $(\mathcal{C} \cap H)/V$  is constant and nonzero.

## 16.1 The Subgroup $2^2.2^{11}.2^{22}.(S_3 \times M_{24})$ (June 2009)

According to the Atlas of Finite Groups [CCN<sup>+</sup>85, p. 234], the Monster group  $M$  has a class of maximal subgroups  $H$  of the type  $2^2.2^{11}.2^{22}.(S_3 \times M_{24})$ . Currently the character table of  $H$  and the class fusion into  $M$  are not available in GAP. We are interested in the permutation character  $1_H^G$ , and we will compute it without this information.

The subgroup  $H$  normalizes a Klein four group  $E$  whose involutions lie in the class 2B. We fix an involution  $z$  in  $E$ , and set  $G = C_M(z)$ ,  $U = C_H(z)$ , and  $V = C_H(E)$ . Further, let  $N$  be the extraspecial normal subgroup of order  $2^{25}$  in  $G$ .

So  $G$  has the structure  $2_+^{1+24}.Co_1$ , and  $U$  has index three in  $H$ . The order of  $NU/N$  is a multiple of  $2^{2+11+22-25} \cdot 2 \cdot |M_{24}|$ , and  $NU/N$  occurs as a subgroup of  $G/N \cong Co_1$ .

```
gap> co1:= CharacterTable( "Co1" );;
gap> order:= 2^(2+11+22-25) * 2 * Size( CharacterTable( "M24" ) );
501397585920
gap> maxes:= List( Maxes( co1 ), CharacterTable );;
gap> filt:= Filtered( maxes, t -> Size( t ) mod order = 0 );
[ CharacterTable( "2^11:M24" ) ]
gap> List( filt, t -> Size( t ) / order );
[ 1 ]
gap> k:= filt[1];;
```

The list of maximal subgroups of  $Co_1$  (see [CCN<sup>+</sup>85, p. 183]) tells us that  $NU/N$  is a maximal subgroup  $K$  of  $Co_1$  and has the structure  $2^{11} : M_{24}$ . In particular,  $U$  contains  $N$  and thus  $U/N \cong K$ .

Let

$$\mathcal{C} = \{g \in M; 3 \nmid |g| \text{ or } 1_V^M(g^3) = 0\}.$$

Then  $(1_H)_{\mathcal{C} \cap H} = (1_U^H - \frac{1}{3}1_V^H)_{\mathcal{C} \cap H}$  holds, as we can see from computations with  $H/V \cong S_3$ , as follows.

```
gap> f:= CharacterTable( "Symmetric", 3 );
CharacterTable( "Sym(3)" )
gap> OrdersClassRepresentatives( f );
[ 1, 2, 3 ]
gap> deg3:= PermChars( f, 3 );
[ Character( CharacterTable( "Sym(3)" ), [ 3, 1, 0 ] ) ]
gap> deg6:= PermChars( f, 6 );
[ Character( CharacterTable( "Sym(3)" ), [ 6, 0, 0 ] ) ]
gap> deg3[1] - 1/3 * deg6[1];
ClassFunction( CharacterTable( "Sym(3)" ), [ 1, 1, 0 ] )
```

The character table of  $G$  is available in GAP, so we can compute the permutation character  $\pi = 1_U^G$  by computing the primitive permutation character  $1_K^{Co_1}$ , identifying it with  $1_{U/N}^{G/N}$ , and then inflating this character to  $G$ .

```
gap> m:= CharacterTable( "M" );
CharacterTable( "M" )
gap> g:= CharacterTable( "MC2B" );
CharacterTable( "2^1+24.Co1" )
gap> pi:= RestrictedClassFunction( TrivialCharacter( k )^co1, g );;
```

Next we consider the permutation character  $\phi = 1_V^G$ . The group  $V$  does not contain  $N$  because  $K$  is perfect. But  $V$  contains  $Z$  because otherwise  $U$  would be a direct product of  $V$  and  $Z$ , which would imply that  $N$  would be a direct product of  $V \cap N$  and  $Z$ . So we can regard  $\phi$  as the inflation of  $1_{V/Z}^{G/Z}$  from  $G/Z$  to  $G$ , i. e., we can perform the computations with the character table of the factor group  $G/Z$ .

```
gap> zclasses:= ClassPositionsOfCentre( g );
gap> gmodz:= g / zclasses;
CharacterTable( "2^1+24.Co1/[ 1, 2 ]" )
gap> invmap:= InverseMap( GetFusionMap( g, gmodz ) );
gap> pibar:= CompositionMaps( pi, invmap );
```

Since  $\phi(g) = [G : V] \cdot |g^G \cap V| / |g^G|$  holds for  $g \in G$ , and since  $g^G \cap V \subseteq g^G \cap VN$ , with equality if  $g$  has odd order, we get  $\phi(g) = 2 \cdot \pi(g)$  if  $g$  has odd order, and  $\phi(g) = 0$  if  $\pi(g) = 0$ .

We want to compute the possible permutation characters with these values.

```
gap> factorders:= OrdersClassRepresentatives( gmodz );
gap> pibar:= [];
gap> for i in [ 1 .. NrConjugacyClasses( gmodz ) ] do
>   if factorders[i] mod 2 = 1 then
>     pibar[i]:= 2 * pibar[i];
>   elif pibar[i] = 0 then
>     pibar[i]:= 0;
>   fi;
> od;
gap> cand:= PermChars( gmodz, rec( torso:= pibar ) );
gap> Length( cand );
1
```

Now we know  $\pi^M = 1_U^M$  and  $\phi^M = 1_V^M$ , so we can write down  $(1_H^M)_c$ .

```
gap> phi:= RestrictedClassFunction( cand[1], g )^m;
gap> pi:= pi^m;
gap> cand:= ShallowCopy( pi - 1/3 * phi );
gap> morders:= OrdersClassRepresentatives( m );
gap> for i in [ 1 .. Length( morders ) ] do
>   if morders[i] mod 3 = 0 and phi[ PowerMap( m, 3 ) [i] ] <> 0 then
>     Unbind( cand[i] );
>   fi;
> od;
```

We claim that  $1_H^M(g) \geq \pi^M(g) - 1/3\psi^M(g)$  for all  $g \in M$ . In order to see this, let  $H'$  denote the index two subgroup of  $H$ , and let  $g \in M$ . Since  $H$  is the disjoint union of  $V$ ,  $H' \setminus V$ , and three  $H$ -conjugates of  $U \setminus V$ , we get

$$\begin{aligned} 1_H^M(g) &= [M : H] \cdot |g^M \cap H| / |g^M| \\ &= [M : H] \cdot ( |g^M \cap V| + 3|g^M \cap U \setminus V| + |g^M \cap H' \setminus V| ) / |g^M| \\ &= [M : H] \cdot ( 3|g^M \cap U| - 2|g^M \cap V| + |g^M \cap H' \setminus V| ) / |g^M| \\ &= 1_U^M(g) - 1/3 \cdot 1_V^G(g) + [M : H] \cdot |g^M \cap H' \setminus V| / |g^M|. \end{aligned}$$

Possible constituents of  $1_H^M$  are those rational irreducible characters of  $M$  that are constituents of  $\pi^M$ .

```
gap> constit:= Filtered( RationalizedMat( Irr( m ) ),
>
> chi -> ScalarProduct( m, chi, pi ) <> 0 );;
```

Now we compute the possible permutation characters that have the prescribed values, are compatible with the given lower bounds for values, and have only constituents in the given list.

```
gap> cand:= PermChars( m,
>   rec( torso:= cand, chars:= constit,
>   lower:= ShallowCopy( pi - 1/3 * phi ),
>   normalsubgroup:= [ 1 .. NrConjugacyClasses( m ) ],
>   nonfaithful:= TrivialCharacter( m ) ) );
[ Character( CharacterTable( "M" ), [ 16009115629875684006343550944921875,
7774182899642733721875, 120168544413337875, 4436049512692980,
215448838605, 131873639625, 760550656275, 110042727795, 943894035,
568854195, 1851609375, 0, 4680311220, 405405, 78624756, 14467005,
178605, 248265, 874650, 0, 76995, 591163, 224055, 34955, 29539, 20727,
0, 0, 375375, 15775, 0, 0, 0, 495, 116532, 3645, 62316, 1017, 11268,
357, 1701, 45, 117, 705, 0, 0, 4410, 1498, 0, 3780, 810, 0, 0, 83, 135,
31, 0, 0, 0, 0, 0, 0, 255, 195, 0, 215, 0, 0, 210, 0, 42, 0, 35, 15,
1, 1, 160, 48, 9, 92, 25, 9, 9, 5, 1, 21, 0, 0, 0, 0, 0, 98, 74, 42, 0,
0, 0, 120, 76, 10, 0, 0, 0, 0, 0, 1, 1, 0, 6, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 5, 3, 0, 0, 0, 18, 0, 10, 0, 3, 3, 0, 1, 1, 1, 1, 0, 0, 2,
0, 0, 0, 0, 0, 0, 2, 0, 0, 0, 0, 0, 6, 12, 0, 0, 2, 0, 0, 0, 2, 0, 0,
1, 1, 0, 0, 0, 0, 0, 0, 2, 0, 2, 0, 0, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0 ] ) ]
```

There is only one candidate, so we have found the permutation character.

## 16.2 The Subgroup $2^3.2^6.2^{12}.2^{18}.(L_3(2) \times 3.S_6)$ (September 2009)

According to the Atlas of Finite Groups [CCN<sup>+</sup>85, p. 234], the Monster group  $M$  has a class of maximal subgroups  $H$  of the type  $2^3.2^6.2^{12}.2^{18}.(L_3(2) \times 3.S_6)$ . Currently the character table of  $H$  and the class fusion into  $M$  are not available in GAP. We are interested in the permutation character  $1_H^G$ , and we will compute it without this information.

The subgroup  $H$  normalizes an elementary abelian group  $E$  of order eight whose involutions lie in the class 2B. We fix an involution  $z$  in  $E$ , and set  $G = C_M(z)$ ,  $U = C_H(z)$ , and  $V = C_H(E)$ . Further, let  $N$  be the extraspecial normal subgroup of order  $2^{25}$  in  $G$ .

So  $G$  has the structure  $2_+^{1+24}.Co_1$ , and  $U$  has index seven in  $H$ . The order of  $NU/N$  is a multiple of  $2^{3+6+12+18-25} \cdot |L_3(2)| \cdot |3.S_6|/7$ , and  $NU/N$  occurs as a subgroup of  $G/N \cong Co_1$ .

```
gap> co1:= CharacterTable( "Co1" );;
gap> order:= 2^(3+6+12+18-25) * 168 * 3 * Factorial( 6 ) / 7;
849346560
gap> maxes:= List( Maxes( co1 ), CharacterTable );;
gap> filt:= Filtered( maxes, t -> Size( t ) mod order = 0 );
[ CharacterTable( "2^(1+8)+.08+(2)" ), CharacterTable( "2^(4+12).(S3xS6)" ) ]
gap> List( filt, t -> Size( t ) / order );
[ 105, 1 ]
gap> o8p2:= CharacterTable( "08+(2)" );;
gap> PermChars( o8p2, rec( torso:= [ 105 ] ) );
[ ]
gap> k:= filt[2];;
```

The list of maximal subgroups of  $Co_1$  (see [CCN<sup>+</sup>85, p. 183]) tells us that  $NU/N$  is a maximal subgroup  $K$  of  $Co_1$  and has the structure  $2^{4+12} \cdot (S_3 \times 3.S_6)$ . (Note that the group  $O_8^+(2)$  has no proper subgroup of index 105.) In particular,  $U$  contains  $N$  and thus  $U/N \cong K$ .

Let  $\mathcal{C}$  be the set of elements in  $M$  whose order is not divisible by 7. Then  $(1_H)_{\mathcal{C} \cap H} = (1_U^H - \frac{1}{3}1_{VN}^H + \frac{1}{21}1_V^H)_{\mathcal{C} \cap H}$  holds, as we can see from computations with  $H/V \cong L_3(2)$ , as follows.

```
gap> f:= CharacterTable( "L3(2)" );
CharacterTable( "L3(2)" )
gap> OrdersClassRepresentatives( f );
[ 1, 2, 3, 4, 7, 7 ]
gap> deg7:= PermChars( f, 7 );
[ Character( CharacterTable( "L3(2)" ), [ 7, 3, 1, 1, 0, 0 ] ) ]
gap> deg42:= PermChars( f, 42 );
[ Character( CharacterTable( "L3(2)" ), [ 42, 2, 0, 2, 0, 0 ] ),
  Character( CharacterTable( "L3(2)" ), [ 42, 6, 0, 0, 0, 0 ] ) ]
gap> deg168:= PermChars( f, 168 );
[ Character( CharacterTable( "L3(2)" ), [ 168, 0, 0, 0, 0, 0 ] ) ]
gap> deg7[1] - 1/3 * deg42[2] + 1/21 * deg168[1];
ClassFunction( CharacterTable( "L3(2)" ), [ 1, 1, 1, 1, 0, 0 ] )
```

(Note that  $VN/V$  is a Klein four group, and there is only one transitive permutation character of  $L_3(2)$  that is induced from such subgroups.)

The character table of  $G$  is available in GAP, so we can compute the permutation character  $\pi = 1_U^G$  by computing the primitive permutation character  $1_K^{Co_1}$ , identifying it with  $1_{U/N}^{G/N}$ , and then inflating this character to  $G$ .

```
gap> m:= CharacterTable( "M" );
CharacterTable( "M" )
gap> g:= CharacterTable( "MC2B" );
CharacterTable( "2^1+24.Co1" )
gap> pi:= RestrictedClassFunction( TrivialCharacter( k )^co1, g );;
```

The permutation character  $\psi = 1_{VN}^G$  can be computed as the inflation of  $1_{VN/N}^{G/N} = (1_{VN/N}^{U/N})^{G/N}$ , where  $1_{VN/N}^{U/N}$  is a character of  $K$  that can be identified with the regular permutation character of  $U/VN \cong S_3$ .

```
gap> nsg:= ClassPositionsOfNormalSubgroups( k );;
gap> nsgsizes:= List( nsg, x -> Sum( SizesConjugacyClasses( k ){ x } ) );;
gap> nn:= nsg[ Position( nsgsizes, Size( k ) / 6 ) ];;
gap> psi:= 0 * [ 1 .. NrConjugacyClasses( k ) ];;
gap> for i in nn do
>   psi[i]:= 6;
> od;
gap> psi:= InducedClassFunction( k, psi, co1 );;
gap> psi:= RestrictedClassFunction( psi, g );;
```

Next we consider the permutation character  $\phi = 1_V^G$ . The group  $V$  does not contain  $N$  because  $K$  does not have a factor group of the type  $S_4$ . But  $V$  contains  $Z$  because  $U/V$  is centerless. So we can regard  $\phi$  as the inflation of  $1_{V/Z}^{G/Z}$  from  $G/Z$  to  $G$ , i. e., we can perform the computations with the character table of the factor group  $G/Z$ .

```
gap> zclasses:= ClassPositionsOfCentre( g );;
gap> gmodz:= g / zclasses;
CharacterTable( "2^1+24.Co1/[ 1, 2 ]" )
gap> invmap:= InverseMap( GetFusionMap( g, gmodz ) );;
gap> psibar:= CompositionMaps( psi, invmap );;
```

Since  $\phi(g) = [G : V] \cdot |g^G \cap V| / |g^G|$  holds for  $g \in G$ , and since  $g^G \cap V \subseteq g^G \cap VN$ , with equality if  $g$  has odd order, we get  $\phi(g) = 4 \cdot \psi(g)$  if  $g$  has odd order, and  $\phi(g) = 0$  if  $\psi(g) = 0$ .

We want to compute the possible permutation characters with these values. This is easier if we “go down” from  $VN$  to  $V$  in two steps.

```
gap> factorders:= OrdersClassRepresentatives( gmodz );;
gap> phibar:= [];;
gap> upperphibar:= [];;
gap> for i in [ 1 .. NrConjugacyClasses( gmodz ) ] do
>   if factorders[i] mod 2 = 1 then
>     phibar[i]:= 2 * psibar[i];
>   elif psibar[i] = 0 then
>     phibar[i]:= 0;
>   else
>     upperphibar[i]:= 2 * psibar[i];
>   fi;
> od;
gap> cand:= PermChars( gmodz, rec( torso:= phibar,
>   upper:= upperphibar,
>   normalsubgroup:= [ 1 .. NrConjugacyClasses( gmodz ) ],
>   nonfaithful:= TrivialCharacter( gmodz ) ) );;
gap> Length( cand );
3
```

One of the candidates computed in this first step is excluded by the fact that it is induced from a subgroup that contains  $N/Z$ .

```
gap> nn:= First( ClassPositionsOfNormalSubgroups( gmodz ),
>   x -> Sum( SizesConjugacyClasses( gmodz ){x} ) = 2^24 );
[ 1 .. 4 ]
gap> cont:= PermCharInfo( gmodz, cand ).contained;;
gap> cand:= cand{ Filtered( [ 1 .. Length( cand ) ],
>   i -> Sum( cont[i]{ nn } ) < 2^24 ) };;
gap> Length( cand );
2
```

Now we run the second step. After excluding the candidates that cannot be induced from subgroups whose intersection with  $N/Z$  has index four in  $N/Z$ , we get four solutions.

```
gap> poss:= [];;
gap> for v in cand do
>   phibar:= [];
>   upperphibar:= [];
>   for i in [ 1 .. NrConjugacyClasses( gmodz ) ] do
>     if factorders[i] mod 2 = 1 then
>       phibar[i]:= 2 * v[i];
>     elif v[i] = 0 then
>       phibar[i]:= 0;
>     else
>       upperphibar[i]:= 2 * v[i];
>     fi;
>   od;
>   Append( poss, PermChars( gmodz, rec( torso:= phibar,
>     upper:= upperphibar,
>     normalsubgroup:= [ 1 .. NrConjugacyClasses( gmodz ) ],
```

```

> nonfaithful:= TrivialCharacter( gmodz ) ) );
> od;
gap> Length( poss );
6
gap> cont:= PermCharInfo( gmodz, poss ).contained;;
gap> poss:= poss{ Filtered( [ 1 .. Length( poss ) ],
> i -> Sum( cont[i]{ nn } ) < 2^23 ) };;
gap> Length( poss );
4
gap> phicand:= RestrictedClassFunctions( poss, g );

```

Since we have several candidates for  $1_V^G$ , we form the linear combinations for all these candidates.

```

gap> phicand:= RestrictedClassFunctions( poss, g );
gap> phicand:= InducedClassFunctions( phicand, m );
gap> psi:= psi^m;;
gap> pi:= pi^m;;
gap> cand:= List( phicand,
> phi -> ShallowCopy( pi - 1/3 * psi + 1/21 * phi ) );
gap> morders:= OrdersClassRepresentatives( m );
gap> for x in cand do
> for i in [ 1 .. Length( morders ) ] do
> if morders[i] mod 7 = 0 then
> Unbind( x[i] );
> fi;
> od;
> od;

```

Exactly one of the candidates has only integral values.

```

gap> cand:= Filtered( cand, x -> ForAll( x, IsInt ) );
[ [ 4050306254358548053604918389065234375, 148844831270071996434375,
2815847622206994375, 14567365753025085, 3447181417680, 659368198125,
3520153823175, 548464353255, 5706077895, 3056566695, 264515625, 0,
19572895485, 6486480, 186109245, 61410960, 758160, 688365,, 172503,
1264351, 376155, 137935, 99127, 52731, 0, 0, 119625, 3625, 0, 0, 0, 0,
402813, 29160, 185301, 2781, 21069, 1932, 4212, 360, 576, 1125, 0, 0,,
, 2160, 810, 0, 0, 111, 179, 43, 0, 0, 0, 0, 0, 0, 0, 185, 105, 0, 65,
0, 0,,,, 0, 0, 0, 0, 337, 105, 36, 157, 37, 18, 18, 16, 4, 21, 0, 0,
0, 0, 0,,,, 0, 0, 60, 40, 10, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0,, 0, 0, 0,
0, 0, 0, 0, 0, 0, 5, 1, 0, 0, 0,,,, 0, 0, 0, 0, 0, 0, 0, 0, 0, 3, 0,
0, 0, 0, 0, 0,, 0, 0, 0, 6, 8, 0, 0, 2, 0, 0, 0, 0, 0, 0, 0, 0,, 0,
0, 0, 0, 0,, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ] ]

```

Possible constituents of  $1_H^M$  are those rational irreducible characters of  $M$  that are constituents of  $\pi^M$ .

```

gap> constit:= Filtered( RationalizedMat( Irr( m ) ),
> chi -> ScalarProduct( m, chi, pi ) <> 0 );

```

Now we compute the possible permutation characters that have the prescribed values and have only constituents in the given list.

```

gap> cand:= PermChars( m, rec( torso:= cand[1], chars:= constit ) );
[ Character( CharacterTable( "M" ), [ 4050306254358548053604918389065234375,

```

```

148844831270071996434375, 2815847622206994375, 14567365753025085,
3447181417680, 659368198125, 3520153823175, 548464353255, 5706077895,
3056566695, 264515625, 0, 19572895485, 6486480, 186109245, 61410960,
758160, 688365, 58310, 0, 172503, 1264351, 376155, 137935, 99127,
52731, 0, 0, 119625, 3625, 0, 0, 0, 0, 402813, 29160, 185301, 2781,
21069, 1932, 4212, 360, 576, 1125, 0, 0, 1302, 294, 0, 2160, 810, 0, 0,
111, 179, 43, 0, 0, 0, 0, 0, 0, 185, 105, 0, 65, 0, 0, 224, 0, 14,
0, 0, 0, 0, 0, 337, 105, 36, 157, 37, 18, 18, 16, 4, 21, 0, 0, 0, 0, 0,
70, 38, 14, 0, 0, 0, 60, 40, 10, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 10, 0,
0, 0, 0, 0, 0, 0, 0, 0, 5, 1, 0, 0, 0, 24, 0, 6, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 3, 0, 0, 0, 0, 0, 2, 0, 0, 0, 0, 0, 6, 8, 0, 0, 2, 0, 0,
0, 0, 0, 0, 0, 2, 0, 0, 0, 0, 0, 4, 0, 2, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 4, 0, 0, 0 ] ) ]

```

There is only one candidate, so we have found the permutation character.

### 16.3 The Subgroup $2^5.2^{10}.2^{20}.(S_3 \times L_5(2))$ (October 2009)

According to the Atlas of Finite Groups [CCN<sup>+</sup>85, p. 234], the Monster group  $M$  has a class of maximal subgroups  $H$  of the type  $2^5.2^{10}.2^{20}.(S_3 \times L_5(2))$ . Currently the character table of  $H$  and the class fusion into  $M$  are not available in GAP. We are interested in the permutation character  $1_H^G$ , and we will compute it without this information.

The subgroup  $H$  normalizes an elementary abelian group  $E$  of order 32 whose involutions lie in the class 2B. We fix an involution  $z$  in  $E$ , and set  $G = C_M(z)$ ,  $U = C_H(z)$ , and  $V = C_H(E)$ . Further, let  $N$  be the extraspecial normal subgroup of order  $2^{25}$  in  $G$ .

So  $G$  has the structure  $2_+^{1+24}.Co_1$ , and  $U$  has index 31 in  $H$ . The order of  $NU/N$  is a multiple of  $2^{5+10+20-25} \cdot |L_5(2)| \cdot |S_3|/31$ , and  $NU/N$  occurs as a subgroup of  $G/N \cong Co_1$ .

```

gap> co1:= CharacterTable( "Co1" );;
gap> order:= 2^35*Size( CharacterTable( "L5(2)" ) ) *6 / 2^25 / 31;
1981808640
gap> maxes:= List( Maxes( co1 ), CharacterTable );;
gap> filt:= Filtered( maxes, t -> Size( t ) mod order = 0 );
[ CharacterTable( "2^11:M24" ), CharacterTable( "2^(1+8)+.08+(2)" ),
  CharacterTable( "2^(2+12):(A8xS3)" ) ]
gap> List( filt, t -> Size( t ) / order );
[ 253, 45, 1 ]
gap> m24:= CharacterTable( "M24" );;
gap> cand:= PermChars( m24, rec( torso:=[ 253 ] ) );
[ Character( CharacterTable( "M24" ), [ 253, 29, 13, 10, 1, 5, 5, 1, 3, 2, 1,
  1, 1, 1, 3, 0, 2, 1, 1, 1, 0, 0, 1, 1, 0, 0 ] ) ]
gap> TestPerm5( m24, cand, m24 mod 11 );
[ ]
gap> PermChars( CharacterTable( "08+(2)" ), rec( torso:=[ 45 ] ) );
[ ]
gap> k:= filt[3];;

```

The list of maximal subgroups of  $Co_1$  (see [CCN<sup>+</sup>85, p. 183]) tells us that  $NU/N$  is a maximal subgroup  $K$  of  $Co_1$  and has the structure  $2^{2+12}.(A_8 \times S_3)$ . (Note that the group  $M_{24}$  has no proper subgroup of index 253, which is shown above using the 11-modular Brauer table of  $M_{24}$ . Furthermore, the group  $O_8^+(2)$  has no subgroup of index 45.) In particular,  $U$  contains  $N$  and thus  $U/N \cong K$ .



Now we transfer this linear combination to the character tables which are given in our situation.

Those constituents that are induced from subgroups of  $H$  above  $W$  can be identified uniquely via their degrees and their values distribution; we compute these characters in the character table of  $U/W$  obtained as a factor table of the character table of  $U/N$ , lift them back to  $U/N$ , induce them to  $G/N$ , inflate them to  $G$ , and then induce them to  $M$ .

```

gap> a8degrees:= List( above{ Filtered( nonzero,
>                                     x -> x <= Length( above ) ) },
>                                     x -> x[1] ) / 31;
[ 1, 8, 15, 28, 56, 56, 70, 105, 120, 168, 336, 336 ]
gap> a8tbl:= subtbl / [ 1, 2 ];;
gap> invtoa8:= InverseMap( GetFusionMap( subtbl, a8tbl ) );
gap> nsg:= ClassPositionsOfNormalSubgroups( k );
gap> nn:= First( nsg, x -> Sum( SizesConjugacyClasses( k ){ x } ) = 6*2^14 );
gap> a8tbl_other:= k / nn;;
gap> g:= CharacterTable( "MC2B" );
CharacterTable( "2^1+24.Co1" )
gap> constit:= [];
gap> for i in [ 1 .. Length( a8degrees ) ] do
>   cand:= PermChars( a8tbl_other, rec( torso:= [ a8degrees[i] ] ) );
>   filt:= Filtered( perms, x -> x^tbl = above[ nonzero[i] ] );
>   filt:= List( filt, x -> CompositionMaps( x, invtoa8 ) );
>   cand:= Filtered( cand,
>                   x -> ForAny( filt, y -> Collected( x ) = Collected(y) ) );
>   Add( constit, List( Induced( Restricted( Induced(
>     Restricted( cand, k ), co1 ), g ), m ), ValuesOfClassFunction ) );
> od;
gap> List( constit, Length );
[ 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1 ]

```

Dealing with the remaining constituents is more involved. For a permutation character  $1_{X/V}^{U/V}$ , we compute  $1_{WX/V}^{U/V}$ , a character whose degree is half as large and which can be regarded as a character of  $U/W$ . This character can be treated like the ones above: We lift it to  $U/N$ , induce it to  $G/N$ , and inflate it to  $G/Z(G)$ ; let this character be  $1_Y^{G/Z(G)}$ , for some subgroup  $Y$ . Then we compute the possible permutation characters of  $G/Z(G)$  that can be induced from a subgroup of index two inside  $Y$ , inflate these characters to  $G$  and then induce them to  $M$ .

```

gap> downdegrees:= List( index2{ Filtered( nonzero,
>                                     x -> x > Length( above ) )
>                                     - Length( above ) },
>                                     x -> x[1] ) / 31;
[ 30, 210, 210, 1920 ]
gap> f:= g / ClassPositionsOfCentre( g );
gap> forders:= OrdersClassRepresentatives( f );
gap> inv:= InverseMap( GetFusionMap( g, f ) );
gap> for j in [ 1 .. Length( downdegrees ) ] do
>   chars:= [];
>   cand:= PermChars( a8tbl_other, rec( torso:= [ downdegrees[j]/2 ] ) );
>   filt:= Filtered( perms, x -> x^tbl = index2[ nonzero[
>     j + Length( a8degrees ) ] - Length( above ) ] );
>   filt:= Induced( subtbl, a8tbl, filt,
>                   GetFusionMap( subtbl, a8tbl ) );
>   cand:= Filtered( cand, x -> ForAny( filt,
>                                     y -> Collected( x ) = Collected( y ) ) );

```

```

> cand:= Restricted( Induced( Restricted( cand, k ), col ), g );
> for chi in cand do
>   cchi:= CompositionMaps( chi, inv );
>   upper:= [];
>   pphi:= [];
>   for i in [ 1 .. NrConjugacyClasses( f ) ] do
>     if forders[i] mod 2 = 1 then
>       pphi[i]:= 2 * cchi[i];
>     elif cchi[i] = 0 then
>       pphi[i]:= 0;
>     else
>       upper[i]:= 2* cchi[i];
>     fi;
>   od;
>   Append( chars, PermChars( f, rec( torso:= ShallowCopy( pphi ),
>     upper:= upper,
>     normalsubgroup:= [ 1 .. 4 ],
>     nonfaithful:= cchi ) ) );
> od;
> Add( constit, List( Induced( Restricted( chars, g ), m ),
>   ValuesOfClassFunction ) );
> od;
gap> List( constit, Length );
[ 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 3, 10, 10, 2 ]

```

Now we form the possible linear combinations.

```

gap> cand:= List( Cartesian( constit ), 1 -> sol * 1 );;
gap> m:= CharacterTable( "M" );
CharacterTable( "M" )
gap> morders:= OrdersClassRepresentatives( m );;
gap> for x in cand do
>   for i in [ 1 .. Length( morders ) ] do
>     if morders[i] mod 31 = 0 or morders[i] mod 21 = 0 then
>       Unbind( x[i] );
>     fi;
>   od;
> od;

```

Exactly one of the candidates has only integral values.

```

gap> cand:= Filtered( cand, x -> ForAll( x, IsInt ) );
[ [ 391965121389536908413379198941796875, 23914487292951376996875,
474163138042468875, 9500455925885925, 646346515815, 334363486275,
954161764875, 147339103275, 1481392395, 1313281515, 0, 8203125,
9827885925, 1216215, 91556325, 9388791, 115911, 587331, 874650, 0,
79515, 581955, 336375, 104371, 62331, 36855, 0, 0, 0, 0, 28125, 525,
1125, 0, 188325, 16767, 88965, 2403, 9477, 1155, 891, 207, 351, 627, 0,
0, 4410, 1498, 0, 0, 0, 30, 150, 91, 151, 31, 0, 0, 0, 0, 0, 0, 0, 0,
0, 125, 0, 5, 5,,,,, 0, 0, 0, 0, 141, 45, 27, 61, 27, 9, 9, 7, 3, 15,
0, 0, 0, 0, 0, 98, 74, 42, 0, 0, 30, 0, 0, 0, 6, 6, 6,,, 1, 1, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0,,,,, 0, 0, 0, 0, 0, 0,
0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 2, 0, 0, 0, 0, 0, 0, 0, 2, 2, 0, 2,,, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,,, 0, 0, 0, 0, 0, 0,,, 0, 0, 0, 0, 0,
0,, 0, 0, 0 ] ]

```

Now we compute the possible permutation characters that have the prescribed values.

```
gap> cand:= PermChars( m, rec( torso:= cand[1] ) );
[ Character( CharacterTable( "M" ), [ 391965121389536908413379198941796875,
23914487292951376996875, 474163138042468875, 9500455925885925,
646346515815, 334363486275, 954161764875, 147339103275, 1481392395,
1313281515, 0, 8203125, 9827885925, 1216215, 91556325, 9388791, 115911,
587331, 874650, 0, 79515, 581955, 336375, 104371, 62331, 36855, 0, 0,
0, 0, 28125, 525, 1125, 0, 188325, 16767, 88965, 2403, 9477, 1155, 891,
207, 351, 627, 0, 0, 4410, 1498, 0, 0, 0, 30, 150, 91, 151, 31, 0, 0,
0, 0, 0, 0, 0, 0, 125, 0, 5, 5, 210, 0, 42, 0, 0, 0, 0, 0, 141, 45,
27, 61, 27, 9, 9, 7, 3, 15, 0, 0, 0, 0, 0, 98, 74, 42, 0, 0, 30, 0, 0,
0, 6, 6, 6, 3, 3, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
1, 1, 0, 18, 0, 10, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0,
2, 0, 0, 0, 0, 0, 0, 2, 2, 0, 2, 3, 3, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 2, 0, 2, 0, 0, 0, 0, 0, 0, 3, 3, 0, 0, 0, 0, 0, 0, 0, 0, 0 ] )
]
```

There is only one candidate, so we have found the permutation character.

## 16.4 The Subgroup $2^{10+16}.O_{10}^+(2)$ (November 2009)

According to the Atlas of Finite Groups [CCN<sup>+</sup>85, p. 234], the Monster group  $M$  has a class of maximal subgroups  $H$  of the type  $2^{10+16}.O_{10}^+(2)$ . Currently the character table of  $H$  and the class fusion into  $M$  are not available in GAP. We are interested in the permutation character  $1_H^G$ , and we will compute it without this information.

The subgroup  $H$  normalizes an elementary abelian group  $E$  of order  $2^{10}$  which contains 496 involutions in the class 2A and 527 involutions in the class 2B. Let  $V$  denote the normal subgroup of order  $2^{26}$  in  $H$ , and set  $\bar{H} = H/N$ . Since the smallest two indices of maximal subgroups of  $\bar{H}$  are 496 and 527, respectively,  $H$  acts transitively on both the 2A and the 2B involutions in  $E$ , and the centralizers of these involutions contain  $V$ .

```
gap> Hbar:= CharacterTable( "010+(2)" );;
gap> U_Abar:= CharacterTable( "010+(2)M1" );
CharacterTable( "S8(2)" )
gap> Index( Hbar, U_Abar );
496
gap> U_Bbar:= CharacterTable( "010+(2)M2" );
CharacterTable( "2^8:08+(2)" )
gap> Index( Hbar, U_Bbar );
527
```

We fix a 2A involution  $z_A$  in  $E$ , and set  $G_A = C_M(z_A)$  and  $U_A = C_H(z_A)$ . So  $G_A$  has the structure  $2.B$  and  $U_A$  has the structure  $2^{10+16}.S_8(2)$ . From the list of maximal subgroups of  $B$  we see that the image of  $G_A$  under the natural epimorphism from  $G_A$  to  $B$  is a maximal subgroup of  $B$  and has the structure  $2^{9+16}.S_8(2)$ .

```
gap> b:= CharacterTable( "B" );
CharacterTable( "B" )
gap> Horder:= 2^26 * Size( Hbar );
1577011055923770163200
gap> order:= Horder / ( 2 * 496 );
1589728887019929600
gap> maxes:= List( Maxes( b ), CharacterTable );;
```

```

gap> filt:= Filtered( maxes, t -> Size( t ) mod order = 0 );
[ CharacterTable( "2^(9+16).S8(2)" ) ]
gap> List( filt, t -> Size( t ) / order );
[ 1 ]
gap> u1:= filt[1];
CharacterTable( "2^(9+16).S8(2)" )

```

Analogously, we fix a 2B involution  $z_B$  in  $E$ , and set  $G_B = C_M(z_B)$  and  $U_B = C_H(z_B)$ , Further, let  $N$  be the extraspecial normal subgroup of order  $2^{25}$  in  $G_B$ . So  $G_B$  has the structure  $2_+^{1+24}.Co_1$ , and  $U_B$  has index 527 in  $G_B$ . From the list of maximal subgroups of  $Co_1$  we see that the image of  $U_B$  under the natural epimorphism from  $G_B$  to  $Co_1$  is a maximal subgroup of  $Co_1$  and has the structure  $2_+^{1+8}.O_8^+(2)$ .

```

gap> co1:= CharacterTable( "Co1" );;
gap> order:= Horder / ( 2^25 * 527 );
89181388800
gap> maxes:= List( Maxes( co1 ), CharacterTable );;
gap> filt:= Filtered( maxes, t -> Size( t ) mod order = 0 );
[ CharacterTable( "2^(1+8).O8+(2)" ) ]
gap> List( filt, t -> Size( t ) / order );
[ 1 ]
gap> u2:= filt[1];
CharacterTable( "2^(1+8).O8+(2)" )

```

First we compute the permutation characters  $\pi_A = 1_{U_A}^M$  and  $\pi_B = 1_{U_B}^M$ .

```

gap> m:= CharacterTable( "M" );
CharacterTable( "M" )
gap> 2b:= CharacterTable( "MC2A" );
CharacterTable( "2.B" )
gap> mm:= CharacterTable( "MC2B" );
CharacterTable( "2^1+24.Co1" )
gap> pi_A:= RestrictedClassFunction( TrivialCharacter( u1 )^b, 2b )^m;;
gap> pi_B:= RestrictedClassFunction( TrivialCharacter( u2 )^co1, mm )^m;;

```

The degree of  $1_H^M$  is of course known.

```

gap> torso:= [ Size( m ) / Horder ];
[ 512372707698741056749515292734375 ]

```

Next we compute some zero values of  $1_H^M$ , using the following conditions.

- For  $g \in M$ , if  $|g|$  does not divide  $|H|$  or if  $|g|$  is not the product of an element order in  $H/V$  and a 2-power. (In fact we could use that the exponent of  $V$  is 4, but this would not improve the result.)
- Let  $U \leq H \leq G$ , and let  $p$  be a prime that does not divide  $[H : U]$ . Then  $U$  contains a Sylow  $p$  subgroup of  $H$ , so each element of order  $p$  in  $H$  is conjugate in  $H$  to an element in  $U$ . For  $g \in G$ ,  $g = g_p h$ , where the order of  $g_p$  is a power of  $p$  such that  $1_U^G(g_p) = 0$  holds, we have  $1_H^G(g) = 0$ . We apply this to  $U \in \{U_A, U_B\}$ .

```

gap> morders:= OrdersClassRepresentatives( m );;
gap> 2parts:= Union( [ 1 ], Filtered( Set( morders ),
> x -> IsPrimePowerInt( x ) and IsEvenInt( x ) ) );
[ 1, 2, 4, 8, 16, 32 ]
gap> factorders:= Set( OrdersClassRepresentatives( Hbar ) );;

```

```

gap> primes_A:= Filtered( Set( Factors( Horder ) ), p -> 496 mod p <> 0 );
[ 3, 5, 7, 17 ]
gap> primes_B:= Filtered( Set( Factors( Horder ) ), p -> 527 mod p <> 0 );
[ 2, 3, 5, 7 ]
gap> primes:= Union( primes_A, primes_B );;
gap> n:= NrConjugacyClasses( m );;
gap> for i in [ 1 .. n ] do
>   if Horder mod morders[i] <> 0 then
>     torso[i]:= 0;
>   elif ForAll( factorders, x -> not morders[i] / x in 2parts ) then
>     torso[i]:= 0;
>   else
>     for p in primes do
>       if morders[i] mod p = 0 then
>         pprime:= morders[i];
>         while pprime mod p = 0 do pprime:= pprime / p; od;
>         pos:= PowerMap( m, pprime )[i];
>         if p in primes_A and pi_A[ pos ] = 0 then
>           torso[i]:= 0;
>         elif p in primes_B and pi_B[ pos ] = 0 then
>           torso[i]:= 0;
>         fi;
>       fi;
>     od;
>   fi;
> od;
gap> torso;
[ 512372707698741056749515292734375,,,,, 0,,,,, 0,, 0,,,,, 0,,
,, 0,,,,, 0, 0, 0,, 0,,, 0,,,,, 0,,,,, 0, 0, 0, 0, 0, 0, 0, 0,,,,
0,,,,, 0, 0, 0, 0, 0, 0, 0,,, 0, 0,,,,, 0,,,,, 0, 0,, 0, 0,,,,, 0, 0, 0, 0,
0,,,,, 0,, 0, 0, 0, 0, 0,, 0, 0, 0, 0, 0, 0,, 0, 0, 0, 0, 0,, 0, 0, 0, 0,
0,,,,, 0,, 0, 0,, 0, 0, 0, 0, 0, 0, 0, 0, 0,, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0 ]

```

Now we want to compute as many nonzero values of  $1_H^M$  as possible, using the same approach as in the previous sections. For that, we first compute several permutation characters  $1_X^M$ , for subgroups  $X$  with the property  $V < X < U_A$  or  $V < X < U_B$ . Then we find several subsets  $\mathcal{C}$  of  $M$ , each being a union of conjugacy classes of  $M$  such that  $(1_H)_{\mathcal{C} \cap H}$  is a linear combination of the characters  $1_X^H$ , restricted to  $\mathcal{C} \cap H$ . This yields the values of  $1_H^M$  on the classes in  $\mathcal{C}$ .

The actual computations are performed with the characters  $1_{X/V}^{H/V}$ . So we build two parallel lists **cand** and **candbar** of permutation characters of  $M$  and of  $H/V$ , respectively. For that, we write two small GAP functions:

- In the function **AddSubgroup0fS82**, we choose a subgroup  $Y$  of  $S_8(2) \cong U_A/V$ , compute  $1_Y^{U_A/V}$ , inflate it to a character of  $U_A$ , induce this character to  $B$ , inflate the result to  $G_A$ , and finally induce this character to  $M$ .
- In the function **AddSubgroup0f08p2**, we choose a subgroup  $Y$  of the factor group  $F \cong O_8^+(2)$  of  $U_B/N$ , compute  $1_Y^F$ , inflate it to a character of  $U_B/N$ , induce this to a character of  $G_B/N \cong Co_1$ , inflate this to a character of  $G_B$ , and finally induce this character to  $M$ .

One difficulty in this case is that choosing a subgroup  $X/V$  of  $H/V$  involves fixing the class fusion into  $H/V$ , but it is not clear which is a compatible class fusion of the corresponding subgroup  $X$  into  $M$ ; therefore, each entry of **cand** is in fact not the permutation character of  $M$  in question but a list of possibilities.

```

gap> cand:= [ [ pi_A ], [ pi_B ] ];
gap> candbar:= [ TrivialCharacter( U_Abar )^Hbar,
>              TrivialCharacter( U_Bbar )^Hbar ];
gap> AddSubgroupOfS82:= function( subname )
>   local psis82;
>
>   psis82:= TrivialCharacter( CharacterTable( subname ) )^U_Abar;
>   Add( cand, [ Restricted( Restricted( psis82, u1 )^b, 2b )^m ] );
>   Add( candbar, psis82 ^ Hbar );
> end;;
gap> tt1:= CharacterTable( "08+(2)" );
CharacterTable( "08+(2)" )
gap> AddSubgroupOf08p2:= function( subname )
>   local psi, list, char;
>
>   psi:= TrivialCharacter( CharacterTable( subname ) )^tt1;
>   list:= [];
>   for char in Orbit( AutomorphismsOfTable( tt1 ), psi, Permuted ) do
>     AddSet( list, Restricted( Restricted( char, u2 ) ^ co1, mm ) ^ m );
>   od;
>   Add( cand, list );
>   Add( candbar, Restricted( psi, U_Bbar ) ^ Hbar );
> end;;

```

Now we choose the subgroups that will turn out to be sufficient for our computations.

```

gap> AddSubgroupOfS82( "08+(2).2" );
gap> AddSubgroupOf08p2( "S6(2)" );
gap> AddSubgroupOfS82( "08-(2).2" );
gap> AddSubgroupOfS82( "A10.2" );
gap> AddSubgroupOfS82( "S4(4).2" );
gap> AddSubgroupOfS82( "L2(17)" );
gap> AddSubgroupOf08p2( "A9" );
gap> AddSubgroupOf08p2( "2^6:A8" );
gap> AddSubgroupOf08p2( "(3xU4(2)):2" );
gap> AddSubgroupOf08p2( "(A5xA5):2^2" );
gap> AddSubgroupOfS82( "S8(2)M4" );

```

In the case of  $A_5 < S_8(2)$ , the function `AddSubgroupOfS82` does not work because there are several class fusions of  $A_5$  into  $S_8(2)$ . We choose one fusion; the fact that it really describes an embedding of an  $A_5$  type subgroup of  $S_8(2)$  can be checked using the function `NrPolyhedralSubgroups`.

```

gap> a5:= CharacterTable( "A5" );
gap> fus:= PossibleClassFusions( a5, U_Abar )[1];
gap> NrPolyhedralSubgroups( U_Abar, fus[2], fus[3], fus[4] );
rec( number := 548352, type := "A5" )
gap> psis82:= Induced( a5, U_Abar, [ TrivialCharacter( a5 ) ], fus )[1];
gap> Add( cand, [ Restricted( Restricted( psis82, u1 )^b, 2b )^m ] );
gap> Add( candbar, psis82 ^ Hbar );
gap> List( cand, Length );
[ 1, 1, 1, 2, 1, 1, 1, 1, 2, 2, 2, 2, 1, 1 ]

```

The following function takes a condition on conjugacy classes in terms of their element orders, which gives a set  $\mathcal{C}$  of elements in  $M$ . It forms the corresponding set of elements in  $H/V$  and tries to express the restriction of  $1_{H/V}$  as a linear combination of the characters  $1_X^{H/V}$  that are stored in the

list `candbar`. If this works and if the corresponding linear combination of the candidates in `cand` is unique, the newly found values of  $1_H^M$  are entered into the list `torso`.

```

gap> Hbarorders:= OrdersClassRepresentatives( Hbar );;
gap> TryCondition:= function( cond )
>   local pos, sol, lincomb, oldknown, i;
>
>   pos:= Filtered( [ 1 .. Length( Hbarorders ) ],
>                   i -> cond( Hbarorders[i] ) );
>   sol:= SolutionMat( candbar{[1..Length(candbar)]}{ pos },
>                     ListWithIdenticalEntries( Length( pos ), 1 ) );
>   if sol = fail then
>     return "no solution";
>   fi;
>
>   pos:= Filtered( [ 1 .. Length( morders ) ], i -> cond( morders[i] ) );
>   lincomb:= Filtered( Set( List( Cartesian( cand ), x -> sol * x ),
>                           x -> ForAll( pos, i -> IsPosInt( x[i] ) or x[i] = 0 ) );
>   if Length( lincomb ) <> 1 then
>     return "solution is not unique";
>   fi;
>
>   lincomb:= lincomb[1];;
>   oldknown:= Number( torso );
>   for i in pos do
>     if IsBound( torso[i] ) then
>       if torso[i] <> lincomb[i] then
>         Error( "contradiction of new and known value at position ", i );
>       fi;
>     elif not IsInt( lincomb[i] ) or lincomb[i] < 0 then
>       Error( "new value at position ", i, " is not a nonneg. integer" );
>     fi;
>     torso[i]:= lincomb[i];
>   od;
>   return Concatenation( "now ", String( Number( torso ) ), " values (",
>                         String( Number( torso ) - oldknown ), " new)" );
> end;;

```

This procedure makes sense only if the elements of  $H$  that satisfy the condition are contained in the full preimage of the classes of  $H/V$  that satisfy the condition. Note that this is in fact the case for the conditions used below. This is clear for condition concerning only **odd** element orders, because  $V$  is a 2-group. Also the set of all elements of the orders 9, 18, and 36 is such a “closed” set, since  $M$  has no elements of order 72. Finally, the set of all elements of the orders 1, 2, and 4 in  $H$  is admissible because it is contained in the preimage of the set of all elements of these orders in  $H/V$ .

```

gap> TryCondition( x -> x mod 7 = 0 and x mod 3 <> 0 );
"now 99 values (7 new)"
gap> TryCondition( x -> x mod 17 = 0 and x mod 3 <> 0 );
"now 102 values (3 new)"
gap> TryCondition( x -> x mod 5 = 0 and x mod 3 <> 0 );
"now 119 values (17 new)"
gap> TryCondition( x -> 4 mod x = 0 );
"now 125 values (6 new)"
gap> TryCondition( x -> 9 mod x = 0 );
"now 129 values (4 new)"

```

```
gap> TryCondition( x -> x in [ 9, 18, 36 ] );
"now 138 values (9 new)"
```

Possible constituents of  $1_H^M$  are those rational irreducible characters of  $M$  that are constituents of  $\pi^M$ .

```
gap> constit:= Filtered( RationalizedMat( Irr( m ) ),
> x -> ScalarProduct( m, x, pi_A ) <> 0
> and ScalarProduct( m, x, pi_B ) <> 0 );;
```

For the missing values, we can provide at least lower bounds, using that  $U \leq H \leq G$  implies  $1_H^G(g) \geq 1_U^G(g)/[H : U] = [G : H] \cdot 1_U^G(g)/1_U^G(1)$ .

```
gap> lower:= [];
gap> Hindex:= Size( m ) / Horder;
512372707698741056749515292734375
gap> for i in [ 1 .. NrConjugacyClasses( m ) ] do
> lower[i]:= Maximum( pi_A[i] / ( pi_A[1] / Hindex ),
> pi_B[i] / ( pi_B[1] / Hindex ) );
> od;
```

Now we compute the possible permutation characters that have the prescribed values, are compatible with the given lower bounds for values, and have only constituents in the given list.

```
gap> PermChars( m, rec( torso:= torso, chars:= constit, lower:= lower,
> normalsubgroup:= [ 1 .. NrConjugacyClasses( m ) ],
> nonfaithful:= TrivialCharacter( m ) ) );
[ Character( CharacterTable( "M" ), [ 512372707698741056749515292734375,
405589064025344574375, 29628786742129575, 658201521662685,
215448838605, 0, 121971774375, 28098354375, 335229607, 108472455,
164587500, 4921875, 2487507165, 2567565, 26157789, 6593805, 398925, 0,
437325, 0, 44983, 234399, 90675, 21391, 41111, 12915, 6561, 6561,
177100, 7660, 6875, 315, 275, 0, 113373, 17901, 57213, 0, 4957, 1197,
909, 301, 397, 0, 0, 0, 3885, 525, 0, 2835, 90, 45, 0, 103, 67, 43, 28,
81, 189, 9, 9, 9, 0, 540, 300, 175, 20, 15, 7, 420, 0, 0, 0, 0, 0, 0,
0, 165, 61, 37, 37, 0, 9, 9, 13, 5, 0, 0, 0, 0, 0, 0, 77, 45, 13, 0, 0,
45, 115, 19, 10, 0, 5, 5, 9, 9, 1, 1, 0, 0, 4, 0, 0, 9, 9, 3, 1, 0, 0,
0, 0, 0, 0, 4, 1, 1, 0, 24, 0, 0, 0, 0, 0, 6, 0, 0, 0, 0, 0, 0, 1, 0,
4, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 3, 3, 1, 1, 2, 0, 3, 3, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0 ] ) ]
```

There is only one candidate, so we have found the permutation character.

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